Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Chern classes of vector bundles with singular connections

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **73** (1982), n.6, p. 207–220.

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<http://www.bdim.eu/item?id=RLINA_1982_8_73_6_207_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1982.

Geometria differenziale. — *Chern classes of vector bundles with singular connections.* Nota di GIUSEPPE DE CECCO [(*)], presentata [(**)] dal Socio straniero A. LICHNEROWICZ.

RIASSUNTO. — Si fa vedere che alcune classi di Chern di fibrati vettoriali complessi possono essere costruite non solo partendo da connessioni C^∞ ma, sotto certe condizioni, anche da connessioni lineari singolari. Nel caso particolare del fibrato tangente possono essere costruite anche a partire da metriche singolari. Viene fatto uso in modo essenziale della L_2–coomologia di de Rham (introdotta da Cheeger e Teleman).

The aim of the present note is to show that some Chern classes of complex vector bundles can be constructed not only via C^∞ connections but also, under certain conditions, via singular linear connections [(1)].

Let M be a differentiable compact Riemannian manifold and N a closed submanifold of M. We denote by $\delta(p)$ the distance of a point of M from N with respect to the Riemannian structure.

If p is close to N, then $\delta(p)$ coincides with the "geodesic distance" from p to N, which behaves like the Euclidean distance.

We call $\tilde{r}(p)$ a suitable extension to all M of the geodesic distance (defined on a neighbourhood of N).

Let E be a C^∞ vector bundle on M with complex fibre.

Choose an arbitrary C^∞ connection ∇ for E and consider the new "connection"

$$\tilde{\nabla} = \nabla + \tilde{r}^\alpha \mathrm{H} \qquad \alpha \in \mathbf{R} - 0$$

where H is a Hom (E , E)-valued one-form on E such that it is bounded on M and its first derivatives are bounded in modulus by r^{-1} C with C constant.

The "connection" $\tilde{\nabla}$ is in general not C^∞; indeed it is singular. In fact if, for instance, $\alpha < 0$, then $\tilde{r}(p)$ diverges for $p \in \mathrm{N}$.

Now, starting from ∇, one can construct, in the usual way, the Chern classes by the Chern-Weil homomorphism.

If one wishes to repeat the above argument for $\tilde{\nabla}$, one then immediately realizes that the Chern forms, constructed from $\tilde{\nabla}$, no longer induce elements of the de Rham cohomology of M. On the other hand, a suitable one is the L_2-de Rham cohomology $\mathrm{H}_d^*(\mathrm{M}\,;\mathbf{C})$, introduced independently by J. Cheegar

(*) This work was carried out in the framework of the activities of the GNSAGA (CNR - Italy).

(**) Nella seduta dell'11 dicembre 1982.

(1) For a different context under different assumption, see D. Lehmann [4].

and N. Teleman, who also proved that the inclusion of the de Rham complex in the L_2-de Rham complex induces an isomorphism at the cohomological level.

Therefore, in the present paper, we first investigate which relationship must exist among α, dim M, dim N and the order of the Chern forms in order that these latter may be cocycles in the L_2-de Rham complex.

Following [1] we introduce the L_2-Chern classes $\tilde{c}_h(E)$, after constructing a Weil homomorphism from the ring of invariant forms to the ring of the L_2-cohomology thus proving (Theorem A) that for some values of h we have

$$\tilde{c}_h(E) = \iota^*(c_h(E))$$

whre $\iota^* : H^*(M ; \mathbf{C}) \to H^*_d(M ; \mathbf{C})$ is the L_2-de Rham-isomorphism.

Then we consider the particular case in which E is the bundle tangent to the almost complex manifold M. If g_p is a hermitian metric of C^∞-class on the fibre E_p, the "metric" defined by

$$\tilde{g}_p = \tilde{r}^\alpha g_p$$

is, in general, singular.

Then, if we denote by ∇ (resp. $\tilde{\nabla}$) the Levi-Civita "connection" associated to g (resp. $\tilde{g}$) the following holds

$$\tilde{\nabla} = \nabla + r^{-1} H$$

hence the issue

$$\tilde{c}_h(TM) = \iota^*(c_h(TM)) \qquad h < (k-2)/4$$

where k is the codimension of N to M. An analogous result can be established for the Pontrjagin classes of M.

I thank N. Teleman for discussions and advice given during the writing of this paper.

1. Preliminaries

(1.1) Let M be a compact differentiable manifold of dimension n. Let us consider a Riemannian metric Γ on M and a PL-structure compatible with the differentiable structure (this is possible because of Whitehead theorem).

The Riemannian metric Γ becomes a "combinatorial Riemannian metric" Γ' in the sense of Teleman [7], relative to any fixed triangulation $\mathscr{T}$, compatible with the PL-structure.

It follows from this (as we soon see) that the space of the L_2-forms of degree r constructed via the Riemannian metric Γ, $L_2^r((M, \Gamma)) \equiv L_r^2(M, \mathscr{D}i\!f\!f)$, coincides with $L_2^r((M, \Gamma')) \equiv L_2^r(M, \mathscr{PL})$ constructed via the combinatorial metric Γ' $(= \Gamma)$.

(1.2) Indeed recall that the space $L_2^r(M, \mathscr{D}iff)$ is, by definition, the completion of the space $C^\infty(M)$ of the complex differential forms on M respect to the norm

$$\| \omega \|^2 = \int_M \omega \wedge * \overline{\omega}$$

where ω is a r-form, $*$ is the Hodge operator with respect to the metric Γ and $\overline{\omega}$ is the (complex) conjugate of ω.

Recall now the definition of $L_r^2(M, \mathscr{PL})$.

$\mathscr{C} = \{\sigma_\alpha^{(s)}\}_{\alpha \in \Lambda}$ being the fixed triangulation of M (where $\sigma_\alpha^{(s)}$ is an arbitrary closed simplex of dimension s with $0 \leq s \leq m$), we denote $S^*(M) = \{S^r(M), d^r\}_{r \in \mathbf{N}}$ the Sullivan complex (2), where $S^r(M)$ consists of all complex exterior PL-forms of degree r on M with differentiable coefficients and $d^r : S^r(M) \to S^{r+1}(M)$ is the usual exterior differentiation on the r-forms.

The restriction ω_α of $\omega \in S^r(M)$ to any maximal simplex $\sigma_\alpha^{(m)} \in \mathscr{C}$ belongs to $S^r(\sigma_\alpha^{(m)})$ and the L_2-norm can be defined in the usual manner:

$$\| \omega_\alpha \|^2 = \int_{\sigma_\alpha^{(m)}} \omega_\alpha \wedge * \overline{\omega}_\alpha$$

$*$ being the star operator with respect to the combinatorial metric $\Gamma' = \{\Gamma'_\alpha\}_{\alpha \in \Lambda}$ (associated to $\mathscr{C}$). Then we can define on $S^r(M)$ the following norm

$$\| \omega \|^2 = \sum_{\alpha \in \Lambda} \| \omega_\alpha \|^2 \qquad \dim \sigma = m$$

which derives from a scalar product. The completion of $S^r(M)$ with respect to such a norm is, by definition, $L_r^2(M, \mathscr{PL})$.

Now $C^\infty(M) \subset S^*(M)$ and it is easy to verify that $S^*(M) \subset L_2^r(M, \mathscr{D}iff)$, hence the assertion

$$L_2^r(M, \mathscr{D}iff) = L_2^r(M, \mathscr{PL})$$

since the norms with respect to which the completion is made coincide on the space of forms $C^\infty(M) \subset S^*(M)$.

Therefore in the following we simply shall denote by $L_2(M)$ the space of the L_2-forms.

(1.3) Let us set moreover, as in [7]

$$\mathscr{D}_d^r = \{\omega \mid \omega \in L_2^r \quad , \quad d\omega \in L_2^{r+1}\} .$$

The complex $\mathscr{D}_d^* = \{\mathscr{D}_d^r, d^r\}_{r \in \mathbf{N}}$ will be named L_2-*de Rham complex* and his homology $H_d^*(M ; \mathbf{C})$ will be called L_2-*cohomology*.

(2) For further details see D. Sullivan [5] and N. Teleman [7].

The natural inclusion map

$$\iota : C^{\infty}(M) \hookrightarrow \mathscr{D}_d^*(M)$$

as we said above, induces an isomorphism (L_2-de Rham theorem)

$$\iota^* : H^*(M ; \mathbf{C}) \rightarrow H_d^*(M : \mathbf{C})$$

$H^*(M ; \mathbf{C})$ being the singular cohomology of M with complex coefficients.

2. The function $\tilde{r}$.

(2.1) Let N be a closed submanifold of M and let $\delta(p) = d(p, N)$ be the distance from p to N induced by the Riemannian structure Γ. It is known that a unique unit-speed geodesic γ goes through any $p \in M$ close to N that intersects N orthogonally at a point q^*. The length of the geodesic contained in γ and joining p with q^* will be called "geodesic distance" from p to N, $r(p)$. Thus r is not defined on all of M but only on a ε-neighbourhood N_ε of N (3), on which $r(p)$ coincides with $\delta(p)$.

Consider now the following C^{∞} monotone function

$$\varphi : \mathbf{R} \rightarrow \mathbf{R}$$

$$\varphi(t) = t \qquad 0 \leq t \leq (1/2)\,\varepsilon$$

$$= 1 \qquad (2/3)\,\varepsilon \leq t.$$

Then $\varphi \circ r$ is identically equal to 1 on the boundary of N_ε, so that it is possible to extend in to M as the function

$$\tilde{r} : M \rightarrow \mathbf{R}$$

defined by

$$\tilde{r}(p) = \varphi \circ r(p) \qquad p \in N_\varepsilon$$
$$= 1 \qquad p \in M - N_\varepsilon . \tag{2.2}$$

(2.3) In order to describe the geometry of M near to N consider a system of coordinates adapted to the submanifold and precisely the Fermi coordinates.

Set $m = \dim M$, $n = \dim N$ and $k = m - n$. Let $e_1, \cdots, e_k$ be orthonormal sections of the normal bundle of N into M defined in a neighbourhood of $q \in N$. Then $\sum_{h=1}^{k} t_h\, e_h(q)$ is a vector in the space $T_q(N)^{\perp} \subset T_q(M)$.

(3) The neighbourhood can be characterized (see A. Gray [3]).

If $(y_1, \cdots, y_n)$ is an arbitrary system of coordinates on N defined in a neighbourhood $W \subset N$ of q, then *Fermi coordinates* are given by

$$x_i \left(\exp_q \left(\sum_{h=1}^{k} t_h e_h(q) \right)\right) = y_i(q) \qquad i = 1, \cdots, n$$

$$x_j \left(\exp_q \left(\sum_{h=1}^{k} t_h e_h(q) \right)\right) = t_j \qquad j = n+1, \cdots, m.$$

Let U be an open set of M such that $W \subset U \cap N$. Thus if $p \in U$ has coordinates $(x_1, \cdots, x_m)$ then one has (4)

$$r(p) = d(p, N) = \sqrt{x_{n+1}^2 + \cdots + x_m^2}\ .$$

One can easily see that

$$\left| \frac{\partial^{|a|} r}{\partial x^a} \right| \leq C_a r^{1-|a|} \tag{2.4}$$

where C_a is a constant and $a = (a_1, \cdots, a_m)$ is a multiindex with $a_1 + \cdots + a_m = |a|$.

On the other hand, for our considerations, the computation up to the second derivates suffices. Explicitly

$$\frac{\partial r}{\partial x_i} = \begin{cases} 0 & i = 1, \cdots, n \\ \dfrac{x_i}{r} & i = n+1, \cdots, m \end{cases}$$

$$\frac{\partial^2 r}{\partial x_i \partial x_j} = \begin{cases} 0 & i \text{ o } j = 1, \cdots, n \\ \dfrac{\delta_{ij}}{r} - \dfrac{x_i x_j}{r^3} & i, j = n+1, \cdots, m \end{cases}$$

(δ_{ij} Kronecker symbol)

from which

$$\left| \frac{\partial r}{\partial x_i} \right| \leq 1$$

$$\left| \frac{\partial^2 r}{\partial x_i \partial x_j} \right| \leq \left| \frac{\delta_{ij}}{r} \right| + \left| \frac{x_i x_j}{r^3} \right| < \frac{1}{r} \left(1 + \left| \frac{x_i}{r} \right| \left| \frac{x_j}{r} \right| \right) \leq \frac{2}{r}\ .$$

3. L_2–CHERN CLASSES

(3.1) Let E be a C^∞ bundle over the C^∞ manifold M with fibre $\mathbf{C}^q$. Denote by $L_2(M) = \sum_r L_2^r(M)$ the graded ring of the L_2–de Rham complex, formed by L_2-forms on M. The differential operator on $L_2(M)$ is denoted by d ([7]).

(4) See A. Gray [3].

If ∇ is an arbitrary C^∞ connection on E, consider

$$\tilde{\nabla} = \nabla + \tilde{r}^\alpha H \qquad \alpha \in \mathbf{R} - 0$$

where $\tilde{r}$ is the function introduced in § 2 and H is a Hom (E, E)-valued 1-form on E such that it is bounded on M and its first derivatives are bounded in modulus by $\tilde{r}^{-1} C$ with C constant.

The "connection" $\tilde{\nabla}$ is in general not C^∞; indeed it is singular.

We will construct in the usual way the Chern forms on E via $\tilde{\nabla}$, *provided that the forms that appear belong to* $L_2(M)$, which thus replaces the ordinary de Rham complex.

More precisely, let $I_h(G)$ be the vector space of the h-forms on the Lie algebra of G symmetric and invariant with respect to $G = GL(q, \mathbf{C})$ and let

$$W_d : I_h(G) \to H_d^*(M; \mathbf{C}) \qquad W_d = \iota^* \circ W$$

the Weil homomorphism respect to the L_2-cohomology, being $W : I_h(G) \to H^*(M; \mathbf{C})$ the usual Weil homomorphism.

If $\tilde{\Omega} = \tilde{\Omega}(E, \tilde{\nabla})$ is the curvature form associated with $\tilde{\nabla}$, consider the invariant polynomials $\varphi_h(\tilde{\Omega})$, defined by setting

$$\det\left(\lambda I + \frac{1}{2\pi i}\tilde{\Omega}\right) = \sum_{h=0}^{q} (-1)^h \varphi_h(\Omega)\, \lambda^{q-h}$$

where

$$\varphi_h(\tilde{\Omega}) = \varphi(\tilde{\Omega}, \cdots, \tilde{\Omega}) \qquad \varphi \in I_h(G)$$

is the *h-th* Chern form.

Then the *h-th Chern class* constructed via $\tilde{\nabla}$ is

(3.2) $$\tilde{c}_h(E, \tilde{\nabla}) = W_d(\varphi_h(\tilde{\Omega})) .$$

After we shall see that

LEMMA A. *If* $\varphi_h(\tilde{\Omega})$ *is the h-th Chern form with respect to* $\tilde{\nabla}$ *and* $\varphi_h(\Omega)$ *is the analogous with respect to* ∇, *then they are* L_2*-cohomological.*

Therefore

$$\tilde{c}_h(E, \tilde{\nabla}) = W_d(\varphi_h(\tilde{\Omega})) = W_d(\varphi_h(\Omega)) = \iota^*(W(\varphi_h(\Omega)) = \iota^*(c_h(E, \nabla)) .$$

Now let ∇' be an other C^∞ connection and

$$\tilde{\nabla}' = \nabla' + \tilde{r}^\alpha H$$

be constructed as above, then from

$$c_h(E, \nabla) = c_h(E, \nabla') = c_h(E)$$

it follows

$$\tilde{c}_h(\mathrm{E}, \tilde{\nabla}) = \tilde{c}_h(\mathrm{E}, \tilde{\nabla}') = \tilde{c}_h(\mathrm{E}).$$

The class $\tilde{c}_h(\mathrm{E})$ is called *h-th* L_2-*Chern class* of the bundle E.

We can now state the main result:

THEOREM A. *Call* $k = m - n$ *the codimension of the submanifold* N *in* M, *the under the stated assumptions,*

$$\tilde{c}(\mathrm{E}) = \iota^*(c_h(\mathrm{E}))$$

for

$$h < \frac{2-k}{4\alpha} \qquad \alpha \leq -1; \qquad h < \frac{k}{2(1-\alpha)} \qquad 0 < \alpha \leq 1$$

$$h < \frac{k+2}{2(1-\alpha)} \qquad -1 \leq \alpha < 0; \qquad \forall h \qquad \alpha \geq 1.$$

4. PROOF OF THEOREM A

(4.1) As seen in (2.3) we can identify an open neighbourhood $\mathrm{V} \subset \mathrm{M}$ of an arbitrary point of the submanifold N with an open set W of $\mathbf{R}^m$, described through the coordinates $(x_1, \cdots, x_m)$, such that $\mathrm{V} \cap \mathrm{N}$ may be identified to an open set of

$$\mathbf{R}^n = \{(x_1, \cdots, x_m) \mid x_{n+1} = x_{n+2} = \cdots = x_m = 0\}.$$

(4.2) Now we shall determine under which conditions on α, m and n, the *h-th* Chern form $\tilde{\varphi}_h = \varphi_h(\Omega)$ may be a cocycle in $L_2(\mathrm{M})$, i.e. $\tilde{\varphi}_h \in L_2(\mathrm{M})$ and $d\tilde{\varphi}_h = 0$ in the sense of distributions.

Notice first that, but for the singular points, the form $\tilde{\varphi}_h$ is C^∞ and satisfies as it is well known, $d\tilde{\varphi}_h = 0$ in the classical sense.

Consider then the expression of $\tilde{\varphi}_h$ on the chart of domain V:

$$\tilde{\varphi}_h = \sum_a \varphi_h^a \, dx^a \qquad a = (a_1, \cdots, a_h);$$

it *suffices* to check when the first partial derivatives $\partial_h^a/\partial x^i$ are L_2 on the whole manifold M, i.e., to check, being $\tilde{u}(x)$ one of the derivatives, when one has

$$\int_{\tilde{\mathrm{U}}} |\tilde{u}(x)|^2 \, \mathrm{d}x < \infty \tag{4.3}$$

for every open set $\tilde{\mathrm{U}}$ relatively compact in $\mathrm{W} \subset \mathbf{R}^m$, which we can suppose bounded, for example of diameter ε.

(4.4) The concept of order of a function, and as a consequence of a form, will be used in an essential way.

Let $p \in M$ belong to the ε-neighbourhood N_ε of N (introduced in § 2) and as usual $r(p) = d(p, N)$. A form ζ is said to be of order ν with respect to N, if all the components of $\zeta(p)/r(p)^\nu$ are bounded when $r(p)$ is infinitesimal. We shall write $\mathrm{ord}_N(\zeta) = \nu$ or simply $\mathrm{ord}(\zeta) = \nu$.

(4.5) In the above mentionned identification we can still denote by $r : \mathbf{R}^m \to [0, \infty)$ the distance function to the plane $\mathbf{R}^n$. Let $y = (y_1, \cdots, y_n)$ be a system of Euclidean coordinates in $\mathbf{R}^n$ and let $(r, s) = (r, s_1, \cdots, s_{k-1})$ be the polar coordinates in the plane $\mathbf{R}^k$, orthogonal to $\mathbf{R}^n$ in $\mathbf{R}^m$.

If $M_k r^{k-1} dr\, ds$ denotes the volume element in $\mathbf{R}^k$, $M_k = $ const., we have

$$\int_{\tilde{U}} |\tilde{u}(x)|^2 \,\mathrm{d}x = M_k \int_{U \times S^{k-1}} \left(\int_0^\varepsilon |\tilde{u}(y, r, s)|^2 r^{k-1} \,\mathrm{d}r \right) \mathrm{d}y \,\mathrm{d}s$$

where $U = \tilde{U} \cap \mathbf{R}^n \subset \mathbf{R}^n$.

If

$$\tilde{u}(y, r, s) = u(y, r, s) + r^\nu v(y, r, s)$$

with $\nu = \mathrm{ord}(\tilde{u} - u)$, one has

$$\int_0^\varepsilon |\tilde{u}|^2 r^{k-1} \,\mathrm{d}r \leq \int_0^\varepsilon |u|^2 r^{k-1} \,\mathrm{d}r + 2 \int_0^\varepsilon |u|\, |v|\, r^{\nu+k-1} \,\mathrm{d}r + \int_0^\varepsilon |v|^2 r^{2\nu+k-1} \,\mathrm{d}r \leq$$

$$\leq C_1 \int_0^\varepsilon r^{k-1} \,\mathrm{d}r + C_2 \int_0^\varepsilon r^{\nu+k-1} \,\mathrm{d}r + C_3 \int_0^\varepsilon r^{2\nu+k-1}$$

whence the conclusion (4.3) if

$$2\nu + k > 0 \tag{4.6}$$

since $k > 0$. By taking into account the value $\nu = \mathrm{ord}(\mathrm{d}\tilde{\varphi}_h - \mathrm{d}\varphi_h)$, which will be calculated in the next section, the theorem is thus completely proved.

5. Estimate of the order of Chern forms

(5.1) If ∇ is a connection on E, denote by (ω_j^i) $(i, j = 1, \cdots, q)$ the matrix of the connection form (1-form) and by (Ω_j^i) the matrix of the curvature form (2-form) associated with ∇. Then

$$\Omega_j^i = \mathrm{d}\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^k$$

which will be written simply

$$\Omega = d\omega + \omega \wedge \omega \tag{5.2}$$

Likewise for the connection $\tilde{\nabla}$.

It follows from the definition on $\tilde{\nabla}$ that

$$\tilde{\omega} = \omega + \tilde{r}^{\alpha} H$$

then

$$\text{ord}\,(\tilde{\omega} - \omega) = \alpha\,.$$

Thus it follows from

$$d\tilde{\omega} = d\omega + (d\tilde{r}^{\alpha})\,H + \tilde{r}^{\alpha}\,dH$$

on account of (2.4), that (5)

$$\text{ord}\,(d\tilde{\omega} - d\omega) = \alpha - 1\,.$$

We premise the following lemma which will be useful in the sequel

(5.4) LEMMA. *Let* $\tilde{A}$ *and* $\tilde{B}$ *be two forms such that* $\tilde{A} = A + \tilde{r}^{\alpha} F$, $\tilde{B} = B + \tilde{r}^{\beta} G$ *with* $\alpha \leq \beta$ *and* A, B, F, G *bounded. Then*

$$\begin{aligned} \text{ord}\,(\tilde{A} \wedge \tilde{B} - A \wedge B) &= \alpha\,, & \beta \geq 0 \\ &= \alpha + \beta\,, & \beta \leq 0\,. \end{aligned}$$

Proof.

$$\tilde{A} \wedge \tilde{B} = (A + \tilde{r}^{\alpha} F) \wedge (B + \tilde{r}^{\beta} G) = A \wedge B + \tilde{r}^{\beta} A \wedge G + \tilde{r}\, F \wedge B + \tilde{r}^{\alpha+\beta} F \wedge G$$

$$\text{ord}\,(\tilde{A} \wedge \tilde{B} - A \wedge B) = \min\,(\alpha, \beta, \alpha + \beta)$$

whence the conclusion.

(5.5) COROLLARY. *If* $\tilde{A} = A + \tilde{r}^{\alpha} F$, *then*

$$\begin{aligned} \text{ord}\,(\underbrace{\tilde{A} \wedge \cdots \wedge \tilde{A}}_{h \text{ times}} - A \wedge \cdots \wedge A) &= \alpha\,, & \alpha \geq 0 \\ &= h\alpha\,, & \alpha \leq 0\,. \end{aligned}$$

(5.6) Thus it follows that

$$\begin{aligned} \text{ord}\,(\tilde{\omega} \wedge \tilde{\omega} - \omega \wedge \omega) &= \alpha\,, & \alpha \geq 0 \\ &= 2\,\alpha\,, & \alpha \leq 0 \end{aligned}$$

(5) Recall that under our assumptions we have $\alpha \neq 0$, so that all the intervals in which α varies belong to $\mathbf{R} - 0$, even though that will not be explicitly mentioned.

whence by (5.2)

$$\operatorname{ord}(\tilde{\Omega} - \Omega) = \alpha - 1\,, \qquad \alpha \geq -1$$
$$= 2\alpha \quad , \qquad \alpha \leq -1\,.$$

(5.7) Consider now

$$\tilde{\phi}_h(\tilde{\Omega}) = \phi(\tilde{\Omega}, \cdots, \tilde{\Omega}) = \Sigma\, \delta^{j_1 \cdots j_h}_{i_1 \cdots i_h}\, \tilde{\Omega}^{i_1}_{j_1} \wedge, \cdots, \wedge\, \tilde{\Omega}^{i_h}_{j_h}$$

briefly

$$\tilde{\phi}_h = \tilde{\Omega} \wedge \cdots \wedge \tilde{\Omega} \qquad h \text{ times}\,.$$

From (5.5) we deduce

$$\operatorname{ord}(\tilde{\phi}_h - \phi_h) = 2h\alpha \quad , \qquad \alpha \leq -1$$
$$= h(\alpha - 1)\,, \qquad -1 \leq \alpha \leq 1$$
$$= \alpha - 1 \quad , \qquad \alpha \geq 1\,.$$

(5.8) We shall now estimate $\operatorname{ord}(d\tilde{\phi}_h - d\phi_h)$. Observe that

$$d\tilde{\phi}_h = d(\tilde{\Omega} \wedge \cdots \wedge \tilde{\Omega}) = (d\tilde{\Omega}) \wedge \tilde{\Omega} \wedge \cdots \wedge \tilde{\Omega} + \cdots + \tilde{\Omega} \wedge \cdots \wedge \tilde{\Omega} \wedge (d\tilde{\Omega})$$

and

$$d\tilde{\Omega} = d\tilde{\omega} \wedge \tilde{\omega} - \tilde{\omega} \wedge d\tilde{\omega}\,.$$

Putting $\tilde{\theta} = \tilde{\Omega} \wedge \cdots \wedge \tilde{\Omega}$ ($h-1$ times) and keeping in mind that

$$\operatorname{ord}(d\tilde{\Omega} - d\Omega) = \alpha - 1\,, \quad \alpha > 0 \qquad \operatorname{ord}(\tilde{\theta} - \theta) = 2(h-1)\alpha \quad , \qquad \alpha \leq -1$$
$$; \qquad = (h-1)(\alpha - 1)\,, \quad -1 \leq \alpha \leq 1$$
$$= 2\alpha - 1\,, \quad \alpha < 0 \qquad = \alpha - 1 \quad , \qquad \alpha \geq 1\,,$$

and finally examining the various cases, by (5.4), one has for $h \geq 1$

$$\operatorname{ord}(d\tilde{\phi}_h - d\phi_h) = 2h\alpha - 1 \qquad \alpha \leq -1 \tag{5.9}$$
$$= h\alpha - h + \alpha \qquad -1 \leq \alpha < 0$$
$$= h(\alpha - 1) \qquad 0 < \alpha \leq 1$$
$$= \alpha - 1 \qquad \alpha \geq 1\,,$$

which is the value we needed to complete the proof.

6. PROOF OF LEMMA A

(6.1) Denote by $\nabla_t = (1-t)\nabla + t\tilde{\nabla}$ with $t \in [0, 1]$ the homotopy between ∇ and $\tilde{\nabla}$ and by Ω_t the corresponding curvature form (6).

From (4.2) it follows

$$\Omega_t = \Omega + \alpha t \tilde{r}^{\alpha-1} \mathrm{H} + t\tilde{r}^{\alpha} \mathrm{dH} + \omega \wedge t\tilde{r}^{\alpha} \mathrm{H} + t\tilde{r}^{\alpha} \mathrm{H} \wedge \omega + t^2 \tilde{r}^{\alpha} \mathrm{H} \wedge \tilde{r}^{\alpha} \mathrm{H}$$

hence for $t \in (0, 1]$ one has

$$\begin{aligned} \operatorname{ord}(\Omega_t - \Omega) &= \alpha - 1 & \alpha \geq -1 \\ &= 2\alpha & \alpha \leq -1 . \end{aligned}$$

(6.2) Consider the $(2h-1)$-forms

$$\psi^i(t) = \varphi\left(\Omega_t, \cdots, \frac{\mathrm{d}}{\mathrm{d}t}\omega_t, \cdots, \Omega_t\right) \qquad i = 1, \cdots, h$$

where the i-th place is $\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \tilde{\omega} - \omega$, ω_t being the connection form associated to ∇_t.

It is easy to see that

$$\begin{aligned} \operatorname{ord}(\psi^i(t)) &= 2h\alpha - \alpha & \alpha \leq -1 \\ &= h(\alpha - 1) + 1 & -1 \leq \alpha \leq 1 \\ &= \alpha & \alpha \geq 1 \end{aligned}$$

also

$$\operatorname{ord}(\tilde{\varphi}_h - \varphi_h) < \operatorname{ord}(\psi^i(t)) \qquad \forall i = 1, \cdots, h \,;\, t \in (0, 1]$$

hence it follows that $\psi^i(t) \in \mathrm{L}_2(\mathrm{M})$ if $\tilde{\varphi}_h \in \mathrm{L}_2(\mathrm{M})$, on account of (4.6).

Similarly one proves that

$$\operatorname{ord}(\mathrm{d}\tilde{\varphi}_h - \mathrm{d}\varphi_h) < \operatorname{ord}(\mathrm{d}\psi^i(t))$$

whence the conclusion that $\mathrm{d}\psi^i(t) \in \mathrm{L}_2(\mathrm{M})$ too, if $\mathrm{d}\tilde{\varphi}_h \in \mathrm{L}_2(\mathrm{M})$.

Then the lemma is proved, remembering that

$$\varphi(\tilde{\Omega}, \cdots, \tilde{\Omega}) - \varphi(\Omega, \cdots, \Omega) = \mathrm{d}\left[\int_0^1 \sum_{1=0}^{h} \psi^i(t)\,\mathrm{d}t\right]$$

and that the integrant is a polynomial in t.

(6) The proof is similar to that in N. Teleman [6].

7. The case of the tangent bundle

(7.1) We consider now the particular case in which M is an almost complex manifold and E is the tangent bundle of M. On every fibre E_p ($p \in M$) it is possible to define a Hermitian metric g_p, induced by the Riemannian structure Γ and invariant by the almost complex structure.

If $\tilde{r}(p)$ is the function introduced above, we consider in E_p the new sesquilinear form

$$\tilde{g}_p = \tilde{r}(p)^{\alpha} g_p \qquad \alpha \in \mathbf{R} - 0$$

which is not, in general, C^{∞}, nay it is singular.

(7.2) We will find a relationship between the Christoffel symbol $\tilde{\Gamma}^k_{ij}$, constructed via $\tilde{g}$, and the symbol Γ^k_{ij}, constructed via g. Setting

$$\tilde{g}_{ij} = \tilde{r}^{\alpha} g_{ij}$$

one has

$$\widetilde{[i, j, s]} = \tfrac{1}{2}(\partial_j \tilde{g}_{js} + \partial_i \tilde{g}_{ij} - \partial_s \tilde{g}_{ij}) = \tilde{r}^{\alpha}[i, j, s] + \\ + \tfrac{1}{2}\alpha \tilde{r}^{\alpha-1}[(\partial_j \tilde{r}) g_{is} + (\partial_i \tilde{r}) g_{js} - (\partial_s \tilde{r}) g_{ij}]$$

where $\partial_h = \partial/\partial x_h$. Because

$$\tilde{g}^{ks} = \tilde{r}^{-\alpha} g^{ks}$$

one has

$$\tilde{\Gamma}^k_{ij} = \tilde{g}^{ks}\,\widetilde{[i, j, s]} = \Gamma^k_{ij} + \tfrac{1}{2}\alpha g^{ks}\,\tilde{r}^{-1}[(\partial_j \tilde{r}) g_{is} + (\partial_i \tilde{r}) g_{js} - (\partial_s \tilde{r}) g_{ij}]$$

hence

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \tilde{r}^{-1} H^k_{ij}$$

where

$$H^k_{ij} = \tfrac{1}{2}\alpha g^{ks}[(\partial_j \tilde{r}) g_{is} + (\partial_i \tilde{r}) g_{js} - (\partial_s \tilde{r}) g_{ij}]$$

is bounded by (2.4) and $|\partial_h H^k_{ij}| < C r^{-1}$ with C constant.

Then for the connection form

$$\tilde{\omega}^k_i = \Gamma^k_{ij}\, dx^j$$

one has

$$\tilde{\omega}^k_i = \omega^k_i + \tilde{r}^{-1} H^k_j$$

where

$$H^k_i = H^k_{ij}\, dx^j .$$

If ∇ (resp. $\tilde{\nabla}$) is the Riemannian connection associated to g (resp. $\tilde{g}$), from (7.3) one has

$$\tilde{\nabla} = \nabla + r^{-1} \mathrm{H}$$

hence from theorem A, by putting $\alpha = -1$, it follows.

THEOREM B. *Let* M *be a compact almost complex manifold and* N *a closed submanifold of* M *of codimension* k. *Call* $\tilde{r} : \mathrm{M} \to \mathbf{R}$ *an extension of the geodesic distance from* $p \in \mathrm{M}$ *to* N (*defined on a neighbourhood of* N). *Let* E = TM *the tangent bundle of* M *and* g_p *an arbitrary* C^∞ *Hermitian metric on* E_p. *The form on* E *defined as*

$$\tilde{g}_p = \tilde{r}(p)^\alpha g_p \qquad \alpha \in \mathbf{R} - 0$$

is generally singular. If $c_h(\mathrm{E})$ (*resp.* $\tilde{c}_h(\mathrm{E})$) *denotes the h-th Chern class, constructed from the Riemannian connection induced by* g (*resp.* $\tilde{g}$), *then*

$$\tilde{c}_h(\mathrm{E}) = \iota^*(c_h(\mathrm{E})) \qquad h < (k-2)/4$$

where ι^* *is the* L_2*-de Rham-isomorphism.*

8. L_2–PONTRJAGIN CLASSES

(8.1) Let M be a differentiable compact Riemannian manifold and let E be its real, tangent bundle.

As in § 7, we consider on every fibre E_p ($p \in \mathrm{M}$) and inner product g_p and the bilinear form

$$\tilde{g}_p = \tilde{r}(p)^\alpha g_p .$$

Then, as in § 3, it is possible to construct the Pontrjagin classes L_2 of M, $\tilde{\mathrm{P}}_h(\mathrm{M}) = \tilde{p}_h(\mathrm{TM})$.

If Ω is the curvature form of the connection ∇, associated to g, then the explicit expression of the Pontrjagin classes is given by

$$\text{(8.2)} \qquad \mathrm{P}_h(\mathrm{M}) = \left[\frac{[(2h)!]^2}{(2^h h!)(2\pi)^{2h}} \sum_{(i)} \theta^{(2h)}_{i_1 \cdots i_{2h}} \wedge \theta^{(2h)}_{i_1 \cdots i_{2h}} \right]$$

where

$$\theta^{(s)}_{i_1 \cdots i_s} = \frac{1}{s!} \sum_{(j)} \delta(i_1, \cdots, i_s ; j_1, \cdots, j_s)\, \Omega_{j_1 j_2} \wedge \cdots \wedge \Omega_{j_{s-1} j_s}$$

s is an even integer and $\delta(i_1, \cdots, i_s ; j_1, \cdots, j_s)$ is the generalised Kronecker symbol.

Then, by § 6, we construct ∇, associated to g, and it turns out that $\operatorname{ord}(\tilde{\Omega} - \Omega) = -2$. Putting

$$\psi_h = \Sigma \theta \wedge \theta$$

one has

$$\operatorname{ord}(d\tilde{\phi}_h - d\psi_h) = -4h - 1$$

and by (4.6)

$$-2(4h+1) + k > 0$$

whence

$$\tilde{P}_h(M) = \iota^*(P_h(M)) \qquad h < (k-2)/8 .$$

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