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Diffeomorphisms constructively associated with mutually diverging spacetimes which allow a natural identification of event points in general relativity. Part I

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Fisica matematica. — *Diffeomorphisms constructively associated with mutually diverging spacetimes which allow a natural identification of event points in general relativity.* Part I. Nota di GAETANO ZAMPIERI (*), presentata (**) dal Corrisp. A. BRESSAN.

RIASSUNTO. — In questo lavoro si dà una definizione di divergenza fra cronotopi della Relatività Generale e si costruisce un criterio per l'identificazione dei punti eventi di cronotopi divergenti che appartengono ad una classe consistente con la presenza di campi elettromagnetici nel vuoto.

1. INTRODUCTION

This work gives a definition of divergence between any two spacetimes of General Relativity (1), and, in a constructive and unique way, associates each ordered pair of mutually diverging spacetimes, in a certain class, with a diffeomorphism from the first spacetime onto the second; thus it gives a natural absolute concept of event point (2)—see ref. [1] or [2] or [3].

However, the most general result of this note is related to a wider class of spacetimes where some event points may have no correspondents in some diverging spacetimes. In connection with this class, the diffeomorphism, associated in this paper with two mutually diverging spacetimes, maps a subset of the first spacetime into the second. This provides a quasi-absolute natural concept of event point—see [1] or [2].

The problem of finding natural (quasi-)absolute concepts of event points in General Relativity was raised by prof. A. Bressan in [2], and this paper puts forward a first solution of it.

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(1) Human actions influence the physical world and, in some sense, destroy its determinism. Thus the possible actions of men—in particular experimenters'—can produce many "mutually diverging" spacetimes.

(2) The way above is constructive in that the axiom of choice is not used, and it is natural in that the reference frame used for the construction, corresponds (in some sense) to the inertial reference frame of the mass-centre in Special Relativity—see N 3. This does not mean that just one natural absolute concept of event point is possible (with respect to a given class of spacetimes).

* * *

In Sect. 2 a divergence relation on the set of spacetimes is defined and is proved to be an equivalence.

Two spacetimes are called mutually diverging if they coincide on some region of type $\mathscr{P}$ (with respect to both spacetimes), i.e. on some open subset such that every past-pointing and past-inextendible causal curve enters into it and, from that moment on, remains within it—see Def. 2.

In Sect. 3 I present some preliminary results (3) which concern a class $\mathfrak{A}$ of spacetimes defined by conditions ensuring the existence of a particular reference frame (4) uniquely associated with each spacetime. This reference frame has a fundamental role in the construction of the aforementioned diffeomorphism, and will be denoted by η for the spacetime S_4—and η^k for S_4^k $(k = 1, 2, \cdots)$.

There exist of course other reference frames uniquely specified in some classes of spacetimes; but the conditions defining these classes are too restrictive for our purposes (5). For example the spacetimes whose gravitational tensor **G** has a unique field of future-pointing timelike unit eigenvectors—which, by the Einstein equation, can be considered at rest with respect to the matter—are not consistent with the electromagnetic field in vacuo (6).

Sec. 4 is the introduction to the second part of this paper constituted by Sections 4 and 5 — see ref. [7].

* * *

The metric $\boldsymbol{g}$ of the typical spacetime $(S_4, \boldsymbol{g})$—briefly S_4—is assumed to be C^∞—see [5] p. 27, [4] p. 58. I also require spacetimes to be *strongly causal*—see ref. [4].

(3) The proofs of the results presented in Sect. 3 are laborious and will appear in the paper " A choice of a global 4-velocity field in General Relativity " by the author, to be printed in « Atti dell'Ist. Veneto di Scienze, Lettere e Arti » (edition made in proofs).

(4) Here a *reference frame* is a (C^∞) global timelike unit and future-pointing vector field—see ref. [5] p. 52.

(5) This does not mean that the reference frame used here is the unique one specified in a sufficiently wide class of spacetimes. In the paper mentioned in fnt (3) the author considers another reference frame too. If we use this, instead of η, we obtain another criterion for identifying the event points (of diverging spacetimes) generally different from the one obtained in this paper.

(6) In fact, by the Einstein equation $\mathbf{G} = 8\pi\mathbf{E}$, at any event point the energymomentum tensor **E** also has a timelike eigenvector which is unique up to nonzero multiples. But this never occurs for the energy-momentum tensor of the electromagnetic field whose eigenvalues are equal in pairs—see e.g. [6] p. 325.

2. Mutually diverging spacetimes

Definition 1. Let S_4 be a spacetime. The set P to be of type $\mathscr{P}$ in S_4—briefly $P \in \mathscr{P}(S_4)$—if (*i*) it is an open subset of S_4, and (*ii*) every past-pointing and past-inextendible causal curve $c : \mathfrak{J} \rightarrow S_4$ [7] enters P, i.e. $c(\mathfrak{J}) \cap P = \varnothing$, and then remain within it, i.e. $\tau_1, \tau_2 \in \mathfrak{J}, \tau_2 > \tau_1$ and $c(\tau_1) \in P$ imply $c(\tau_2) \in P$.

The *globally hyperbolic* case is particularly interesting [8]. If S_4 is globally hyperbolic and $C \subset S_4$ is compact, then the complement of the causal future $J^+(C)$ of C is a region of type $\mathscr{P}$ in S_4 [9] [10].

Definition 2. The spacetime S_4^1 is said to diverge from the spacetime S_4^2 if there exists $P^{21} \in \mathscr{P}(S_4^k)$ $(k = 1, 2)$ such that the structures of the two spacetimes coincide on P^{21}.

E.g. if the globally hyperbolic spacetimes S_4^k $(k = 1, 2)$ contain a same spacetime S_4 [11] such that $S_4 = S_4^k - J^+(C^k)$ for some compact set $C^k \subset S_4^k$ $(k = 1, 2)$, then S_4^1 diverges from S_4^2.

Proposition 1. *The divergence relation introduced in Def. 2, is an equivalence.*

Proof. Symmetry and reflexivity are trivial.

Let $P^{21} \in \mathscr{P}(S_4^k)$ $(k = 1, 2)$, $P^{32} \in \mathscr{P}(S_4^i)$ $(i = 2, 3)$, and let the structure of $S_4^1\,[S_4^2]$ agree with the one of $S_4^2\,[S_4^3]$ on $P^{21}\,[P^{32}]$. Furthermore let $P^{31} = P^{32} \cap P^{21}$. P^{31} is an open subset of S_4^1 and S_4^3, and (the structures of) S_4^1 and S_4^3 coincide on P^{31}.

(7) $\mathfrak{J}$ is an interval of **R**.

The aforesaid curve is *past-inextendible* if it has no past endpoint. (A point $\mathscr{E}$ is said to be a past endpoint of it, if, for every neighbourhood V of $\mathscr{E}$, there exists $\bar{\tau} \in \mathfrak{J}$ such that $c(\tau) \in V$ for every $\tau \in \mathfrak{J}$ with $\tau \geq \bar{\tau}$—see ref. [4] p. 184).

(8) The spacetime S_4 is said to be *globally hyperbolic* if (it is strongly causal and) for any two points $\mathscr{E}_1, \mathscr{E}_2 \in S_4$, $J^+(\mathscr{E}_1) \cap J^-(\mathscr{E}_2)$ is compact—where $J^+(\mathscr{E})\,[J^-(\mathscr{E})]$ is the causal future [past] of $\{\mathscr{E}\}$. For the definition of the causal future $J^+(A)$ of a set A see e.g. [4] p. 183.

(9) In fact $J^+(C)$ is closed—see [4] p. 207. Furthermore, since $J^-(\mathscr{E}) \cap J^+(C)$ is compact for any $\mathscr{E}$, then every past-pointing and past-inextendible causal curve must enters $S_4 - J^+(C)$—see [4] p. 195—and obviously it cannot leave it (by the definition of the causal future).

(10) Instead, the chronological past—see e.g. [4] p. 183—of a spacelike hypersurface with no edge—see [4] p. 202—generally is not of type $\mathscr{P}$ even in the Minkowski spacetime. In the globally hyperbolic case it is of type $\mathscr{P}$ if we require to the hypersurface to be a Cauchy surface—see [4] p. 205.

(11) The spacetime $(S_4^1, \boldsymbol{g}^1)$ *contains* $(S_4, \boldsymbol{g})$ if (*i*) S_4 is an open submanifold of S_4^1, (*ii*) $\boldsymbol{g}^1|_{S_4} = \boldsymbol{g}$, and (*iii*) $(S_4, \boldsymbol{g})$ has the induced orientation and time orientation—see [5] p. 28.

Now let us prove that $P^{31} \in \mathscr{P}(S_4^1)$. In the same way we can see that $P^{31} \in \mathscr{P}(S_4^3)$ and this completes the proof.

Let $c : \mathfrak{J} \to S_4^1$ be a past-pointing and past-inextendible causal curve— briefly $c \in \Gamma(S_4^1)$. Since $P^{21} \in \mathscr{P}(S_4^1)$, then

(1) $$c(\mathfrak{J}) \cap P^{21} \neq \varnothing ,$$

and

(2) $$\tau_1, \tau_2 \in \mathfrak{J}, \tau_2 > \tau_1, c(\tau_1) \in P^{21} \Rightarrow c(\tau_2) \in P^{21} .$$

From (1)

(3) $$\tilde{\mathfrak{J}} = c^{-1}(P^{21}) \neq \varnothing .$$

Furthermore (2) yields

(4) $$\tau_1 \in \tilde{\mathfrak{J}}, \tau_2 \in \mathfrak{J}, \tau_2 > \tau_1 \Rightarrow \tau_2 \in \tilde{\mathfrak{J}} .$$

Since (*i*) $P^{21} \subset S_4^2$, (*ii*) $\tilde{\mathfrak{J}}$ is a non-empty interval of **R**—from (4)—, and (*iii*) S_4^2 and S_4^1 coincide on P^{21}, then we can define a past-pointing causal curve of S_4^2 in the following way. $\tilde{c} : \tilde{\mathfrak{J}} \to S_4^2, \tau \mapsto c(\tau)$; and we have

(5) $$\tilde{c}(\tilde{\mathfrak{J}}) \subset P^{21} .$$

The curve $\tilde{c}$ is also past-inextendible. In fact, let us suppose the contrary. Then we can extend it to a curve in $\Gamma(S_4^2)$ (12) which remains in P^{21} because $P^{21} \in \mathscr{P}(S_4^2)$. This contradicts the past-inextendibility of c because $P^{21} \subset S_4^1$, and S_4^1 and S_4^2 coincide on P^{21}. Thus $\tilde{c} \in \Gamma(S_4^2)$, and since $P^{32} \in \mathscr{P}(S_4^2)$ then

(6) $$\tilde{c}(\tilde{\mathfrak{J}}) \subset P^{32} \neq \varnothing ,$$

and

(7) $$\tau_1, \tau_2 \in \tilde{\mathfrak{J}}, \tau_2 > \tau_1, \tilde{c}(\tau_1) \in P^{32} \Rightarrow \tilde{c}(\tau_2) \in P^{32} .$$

Now, using (5) and $P^{31} = P^{21} \cap P^{32}$,

$$c(\mathfrak{J}) \cap P^{31} \supset \tilde{c}(\tilde{\mathfrak{J}}) \cap P^{31} = \tilde{c}(\tilde{\mathfrak{J}}) \cap P^{32} .$$

From this inclusion and (6) we obtain $c(\mathfrak{J}) \cap P^{31} \neq \varnothing$ which is a part of the expected result.

In the sequel of the proof let us suppose that $\tau_1, \tau_2 \in \mathfrak{J}, \tau_2 > \tau_1$, $c(\tau_1) \in P^{31}$. We must prove that $c(\tau_2) \in P^{31} = P^{21} \cap P^{32}$. $c(\tau_2) \in P^{21}$ immediately

(12) I mean that there exists $\bar{c} \in \Gamma(S_4^2)$, $\bar{c} : \bar{\mathfrak{J}} \to S_4^2$, whose restriction to an interval $\hat{\mathfrak{J}}$, with $\inf \hat{\mathfrak{J}} = \inf \bar{\mathfrak{J}}$ and $\sup \hat{\mathfrak{J}} < \sup \bar{\mathfrak{J}}$, can be obtained from $\tilde{c}$ by a monotonic parameter change.

comes from (2) and $P^{31} \subset P^{21}$. In order to prove $c(\tau_2) \in P^{32}$ note that the definition of $\tilde{\mathfrak{J}}$ and the inclusion $P^{31} \subset P^{21}$ yield $\tau_1 \in \tilde{\mathfrak{J}}$. From (4) we then have $\tau_2 \in \tilde{\mathfrak{J}}$. We obtained $\tau_1, \tau_2 \in \tilde{\mathfrak{J}}$, $\tau_2 > \tau_1$, $\tilde{c}(\tau_1) \in P^{31}$.

This implies that $\tilde{c}(\tau_2) \in P^{32}$ because of (7) and the inclusion $P^{31} \subset P^{32}$. Now $c(\tau_2) = \tilde{c}(\tau_2) \in P^{32}$. q.e.d.

3. The reference frame

In the sequel I present some results whose proofs will appear in the paper mentioned in fnt (3).

A (strongly causal) spacetime S_4 is said to belong to the class $\mathfrak{G}$ if, for every $\mathscr{E} \in S_4$, one of its neighbourhoods V is such that the union (of the ranges) of the lightlike past-pointing geodesics $\gamma : [0, 1] \to S_4$, with $\gamma(0) \in V$ and $\mathbf{G}[\gamma(1)] \neq 0$ has a compact closure—where $\mathbf{G}$ is the gravitational (covariant) tensor.

On each $S_4 \in \mathfrak{G}$ a vector field ξ, defined as follows, is proved to exist and to be C^∞ (if the metric tensor is C^∞).

Let $\mathscr{E} \in S_4 \in \mathfrak{G}$ and let $Domexp_{\mathscr{E}}$ be the domain of the exponential map at $\mathscr{E}$—see e.g. [4] p. 33—, that is the set of all tangent vectors v at $\mathscr{E}$ such that the maximal geodesic $\gamma_{(\mathscr{E},v)}$, with $(\mathscr{E}, v)$ as initial conditions, has 1 in its domain (13). Then consider (*i*) $t \in \mathscr{L}^-(\mathscr{E}) \cap Domexp_{\mathscr{E}}$, where $\mathscr{L}^-(\mathscr{E})$ is the past lightcone in the tangent space $T(\mathscr{E})$ at $\mathscr{E}$, (*ii*) the (covariant) tensor obtained at $\mathscr{E}$ by parallel transport of $\mathbf{G}[\gamma_{(\mathscr{E},t)}(1)]$ along $\gamma_{(\mathscr{E},t)}$, and (*iii*) the linear map $\varphi_{(\mathscr{E},t)} : T(\mathscr{E}) \to T(\mathscr{E})$ equivalent to it by the metric (14) — see [5] p. 17. Now, by integrating the function $t \mapsto \varphi_{(\mathscr{E},t)}(t)$ from $\mathscr{L}^-(\mathscr{E}) \cap Domexp_{\mathscr{E}}$ to $T(\mathscr{E})$, we can obtain a vector $\xi(\mathscr{E}) \in T(\mathscr{E})$ and therefore a vector field $\xi : \mathscr{E} \mapsto \xi(\mathscr{E})$.

Now, let us consider some conditions which guarantee the timelike character of ξ.

Condition 1. $\mathbf{G}(v, v) \geq 0$, *and the vector* $\tilde{\mathbf{G}}(v)$ (15) *is causal or vanishes, for any choice of* $(\mathscr{E}, v)$ *among the causal points of the tangent bundle.*

(13) For the geodesics (which must be unbroken in this paper) only affine parameters are considered.

The geodesic $\gamma_{(\mathscr{E},v)}$ is a map $\mathfrak{J}_{(\mathscr{E},v)} \to S_4$, where $\mathfrak{J}_{(\mathscr{E},v)}$ is an open interval of $\mathbf{R}$ containing zero. If the equations $x^i = \hat{x}^i(\tau)$ — $i = 1, \cdots, 4$—represent the geodesic $\gamma_{(\mathscr{E},v)}$ (in a neighbourhood of $\mathscr{E}$) with respect to some chart (x^i) at $\mathscr{E}$, then $x^i(\mathscr{E}) = \hat{x}^i(0)$ and $(d\hat{x}^i/d\tau)_{\tau=0} = v^i$, where v^i are the components of v with respect to the basis associated with (x^i).

(14) In terms of components, if (at $\mathscr{E}$) B_{ab}, v^c, and g^{ed} are the components of the preceding covariant tensor, a vector v, and the contravariant tensor associated with the metric tensor respectively, then $g^{ac} B_{cb} v^b$ are those of $\varphi_{(\mathscr{E},t)}(v)$.

CONDITION 2. *Let* N *be the interior of the set of all events at which* **G** *is normal* (16). *Then for any* $\mathcal{E} \in S_4$ *there exists a lightlike past-pointing geodesic* $\gamma : [0, 1] \rightarrow S_4$ *with* $\gamma(0) = \mathcal{E}$ *and* $\gamma(1) \in N$.

If these conditions hold, then ξ is timelike and we can consider the reference frame η associated with it—see fnt (4)

DEFINITION 3. The spacetime S_4 is said to belong to $\mathfrak{A}$ if $S_4 \in \mathfrak{G}$ and Conditions 1 and 2 hold for it, so that the (C^∞) reference frame η is defined on it.

Remark 1. The aforementioned conditions can be physically interpreted in terms of the energy-momentum tensor **E** which is related to **G** by the Einstein equation $\mathbf{G} = 8\,\pi\mathbf{E}$.

Remark 2. In the Minkowski spacetime $\xi(\mathcal{E})$ coincides with the zero vector. However, if we consider Special Relativity and we substitute **E** for **G** in the preceding arguments, then we can see that (*i*) $\xi(\mathcal{E})$ is the flux of 4-momentum across the past lightcone, in the Minkowski spacetime, with vertex at $\mathcal{E}$ (17), and (*ii*) η is parallel (i.e. its covariant derivative vanishes).

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(15) In terms of components, if (at $\mathcal{E}$) G_{ab}, v^c, and g^{ed} are the components of **G**, the vector v, and the contravariant tensor associated with the metric tensor respectively, then $\mathbf{G}(v, v) = G_{ab}\, v^a v^b$ and $g^{ab}\, G_{bc}\, v^c$ are the components of $\tilde{\mathbf{G}}(v)$. The notations $\mathbf{G}(v, v)$ and $\tilde{\mathbf{G}}(v)$ are used in [5]—see [5] pp. 3, 17, 18.

(16) I say that **G** is *normal* at $\mathcal{E} \in S_4$ iff $\tilde{\mathbf{G}}(v)$ is timelike for any causal $v \in T(\mathcal{E})$. This definition is analogous to the one in ref. [5] p. 105 where the energy-momentum tensor **E** (instead of **G**) is considered.

(17) See ref. [6], pp. 282, 383, and 430.