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Precession of the perihelion within a generalized theory for the two body problem

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Meccanica. — *Precession of the perihelion within a generalized theory for the two body problem* (*). Nota di FRANCO CARDIN (**), presentata (***) dal Corrisp. A. BRESSAN.

RIASSUNTO. — Sulla base di una teoria generalizzata di Meccanica Classica per il problema dei Due Corpi, recentemente formulata dall'autore, si considera la questione della precessione del perielio dei pianeti, assente nel caso Newtoniano. Si mostra come la descrizione di questo fenomeno in tale teoria generalizzata è sostanzialmente equivalente a quella offerta dalla Relatività Generale.

1. INTRODUCTION

In a recent work [2] I stated a generalized theory $\mathscr{T}^*$ of Classical Mechanics for the Two Body problem in which

(*i*) a weaker axiom for mass than Mach's one is proposed, which implies the rejection of the (full) classical Action and Reaction principle, and

(*ii*) the well-known Two Body reference frames ($\mathscr{R}_{\mathscr{T}\mathscr{B}}$), rotationless with respect to inertial frames ($\mathscr{R}_{\mathscr{I}}$), have a certain privileged role, in that the existence of a generalized energy integral is postulated within these frames.

The usual theory $\mathscr{T}$ of Classical Mechanics is a particular case of $\mathscr{T}^*$.

As announced in [2], in the present paper—see N. 3—I study within the frame-work of $\mathscr{T}^*$ a natural explanation of the possible precessions for the apsidal points of the orbit of a particle with respect to the other one, at which the origin of a $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ is placed. This explanation of the precession of the perihelion is very close to the well known one that is deduced within General Relativity, in the Schwarzschild universe.

This work is a trial of explaining a physical phenomenon presently described satisfactorily only by General Relativity, on the basis of a classical conception of the physical world; the interest in these trials has been recently increased e.g. by the works of D. Galletto and his co-workers in Torino on the cosmological field.

The present note is self-consistent on the basis of the theory $\mathscr{T}^*$, an outline of which can be found in N. 2.

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2. Some results from the theory of generalized Classical Mechanics $\mathscr{C}^*$

We consider classical physics and regard the motions of *inertial spaces* and *inertial frames* as known. Let $\mathscr{R}_{\mathscr{I}}$ be such a frame. We assume that only the particles S (: " sun ", P_1 in [2]) and P (: " planet ", P_2 in [2]) exist, so that they constitute an *isolated system.*

Here I list briefly the set of axioms for $\mathscr{C}^*$. The axioms substantially different from the usual ones are (*i*) a weaker axiom for mass than Mach's one, cf. A 2 in [2], and (*ii*) an axiom of existence of a generalized energy integral, cf. A 5 in [2]. The axiom of Physical Possibility, A 1 in [2], is omitted, since I will not employ it here.

Weak Mass Axiom. *For some* $\mathrm{M}, m \in \mathbf{R}^+$, *if* x_{M}, x_m $[\boldsymbol{a}_{\mathrm{M}}, \boldsymbol{a}_m]$ *are the position* [*accelerations*] *of* S *and* P *in the inertial frame* $\mathscr{R}_{\mathscr{I}}$ *at some instant t, then we necessarily have*

$$(2.1) \qquad (\mathrm{M}\boldsymbol{a}_{\mathrm{M}} + m\boldsymbol{a}_m) \times (\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{x}_m) = 0\,.$$

Note that the Mach's axiom is obtained by substituting (2.1) with:

$$(2.1)' \qquad \mathrm{M}\boldsymbol{a}_{\mathrm{M}} + m\boldsymbol{a}_m = \mathbf{0}\,.$$

Dynamical law. *In connection with an arbitrary choice of* $\mathscr{R}_{\mathscr{I}}$ *there is a function* (*force*) f *such that if, at the instant* t, S *and* P *have the positions* $\boldsymbol{x}_{\mathrm{M}}$ *and* $\boldsymbol{x}_m$, *and velocities* $\boldsymbol{v}_{\mathrm{M}}$ *and* $\boldsymbol{v}_m$ *respectively then*

$$(2.2) \qquad \mathrm{M}\boldsymbol{a}_{\mathrm{M}} = \boldsymbol{f}(\boldsymbol{x}_m, \boldsymbol{x}_{\mathrm{M}}, \boldsymbol{v}_m, \boldsymbol{v}_{\mathrm{M}}) \quad , \quad m\boldsymbol{a}_m = f(\boldsymbol{x}_{\mathrm{M}}, \boldsymbol{x}_m, \boldsymbol{v}_{\mathrm{M}}, \boldsymbol{v}_m)\,.$$

Homogeneous and isotropic properties of inertial spaces are included in the following axiom, where $\mathscr{SO}(n)$ denotes the proper orthogonal groups on $\mathbf{R}^n$.

Homogeneity and Isotropy. *There is a function* $\mathbf{F} : [\mathbf{R}^3 \diagdown \{\mathbf{0}\}] \times \mathbf{R}^3 \to \mathbf{R}^3$ *for which*

$$(2.3) \qquad \mathbf{F}(\boldsymbol{x}_m - \boldsymbol{x}_{\mathrm{M}}, \boldsymbol{v}_m - \boldsymbol{v}_{\mathrm{M}}) = \boldsymbol{f}(\boldsymbol{x}_{\mathrm{M}}, \boldsymbol{x}_m, \boldsymbol{v}_{\mathrm{M}}, \boldsymbol{v}_m)\,,$$

where $\boldsymbol{f}$ *is as in* (2.2), *and, for all* $(\boldsymbol{u}, \boldsymbol{w}) \in [\mathbf{R}^3 \diagdown \{\mathbf{0}\}] \times \mathbf{R}^3$ *and all* $\mathrm{Q} \in \mathscr{SO}(3)$

$$(2.4) \qquad \mathrm{Q}\mathbf{F}(\boldsymbol{u}, \boldsymbol{w}) = \mathbf{F}(\mathrm{Q}\boldsymbol{u}, \mathrm{Q}\boldsymbol{w})\,.$$

By a theorem of Cauchy we obtain the following representation for $\mathbf{F}$:

$$(2.5) \qquad \mathbf{F}(\boldsymbol{u}, \boldsymbol{w}) = \mathscr{A}\boldsymbol{u} + \mathscr{B}\boldsymbol{w} + \mathscr{C}\boldsymbol{u} \wedge \boldsymbol{w}\,,$$

where $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ are functions of $|\boldsymbol{u}|, |\boldsymbol{w}|$, and $\boldsymbol{u} \times \boldsymbol{w}$. If we assume the validity of (2.1)′ then for $\mathbf{F}$ the following (restricted) representation holds:

$$(2.5)' \qquad \mathbf{F}(\boldsymbol{u}, \boldsymbol{w}) = \mathscr{A}\boldsymbol{u} + \mathscr{B}\boldsymbol{w} \qquad (\mathscr{C} \equiv 0)\,.$$

My intention is to make use of the full representation (2.5) for $\mathbf{F}$. Mach's axiom has been generalized in order to rendeı this use possible. From a physical point of view the choice (2.1) leads us to reject, in a classical (non relativistic) framework, the full Action and Reaction principle.

In the following I denote by $\mathcal{R}_{\mathcal{T}\mathcal{B}}$ (Two Body frame) a reference frame whose origin is (always) in S and which is rotationless with respect to inertial spaces. Set

$$(2.6) \qquad \boldsymbol{q} = \boldsymbol{x}_m - \boldsymbol{x}_{\mathrm{M}} \quad , \quad m^* = \frac{\mathrm{M}m}{\mathrm{M}+m} \qquad (\textit{reduced mass}) .$$

By the following axiom the frames $\mathcal{R}_{\mathcal{T}\mathcal{B}}$ are, in a some sense, privileged.

ENERGETIC AXIOM *If* $\mathcal{R}_{\mathcal{T}\mathcal{B}}$ *has the origin in* S *and has the same orthonormal basis* $\{e_h\}_{h=1,2,3}$ *as* $\mathcal{R}_{\mathcal{I}}$, *the function* $\mathfrak{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, *for which the motion of* P *necessarily fulfils the equation*

$$(2.7) \qquad m^* \ddot{\boldsymbol{q}} = \mathfrak{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) ,$$

is afforded by a generalized potential $\mathrm{V} \in \mathscr{C}^{(2)}([\mathbf{R}^3 \setminus \mathrm{A}] \times \mathbf{R}^3 ; \mathbf{R})$ *for some set* A *without internal points, such that the functions (components of* $\mathfrak{F}$)

$$(2.8) \qquad \mathfrak{F}_h = \frac{\partial \mathrm{V}}{\partial q_h} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathrm{V}}{\partial \dot{q}_h} \quad , \quad \mathfrak{F}_h \in \mathscr{C}([\mathbf{R}^3 \setminus \mathrm{A}] \times \mathbf{R}^3 ; \mathbf{R}) , \qquad h = 1, 2, 3 ,$$

have continuous extensions onto $[\mathbf{R}^3 \setminus \{\mathbf{0}\}] \times \mathbf{R}^3$.

Note that if (2.1)′ holds, instead of (2.1), then so does

$$(2.9) \qquad \mathfrak{F} = \mathbf{F} .$$

In connection with a choice of $\mathcal{R}_{\mathcal{T}\mathcal{B}}$ let us set

$$(2.10) \qquad \mathbf{L}^* = m^* \boldsymbol{q} \wedge \dot{\boldsymbol{q}} \qquad (\textit{reduced moment of momentum}),$$

and let ω^{A} be the angular velocity of the plane $(\mathrm{S}, \hat{\mathrm{L}}^*, \hat{q})$ with respect to $\mathcal{R}_{\mathcal{T}\mathcal{B}}$, where $\hat{\mathrm{L}}^* = \mathbf{L}^*/|\mathbf{L}^*|$, $\hat{q} = \boldsymbol{q}/q$, $q = |\boldsymbol{q}|$, and $\hat{\omega}^{\mathrm{A}} = \omega^{\mathrm{A}}/|\omega^{\mathrm{A}}|$.

The following theorem is proved in [2].

THEOREM 1. *According to the theory* $\mathcal{T}^*$, *based on the above axioms,* (i) *the most general force function* $\mathfrak{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ $[\mathbf{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})]$ *relative to* $\mathcal{R}_{\mathcal{T}\mathcal{B}}[\mathcal{R}_{\mathcal{I}}]$ *is expressed by* (2.11) [(2.12)] *below for some integration constant* $\mathrm{K} \in \mathbf{R}$ *and for some function* $\mathrm{U}(q) \in \mathscr{C}^{(2)}$:

$$(2.11) \qquad \mathfrak{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{\mathrm{K}}{q^3} \boldsymbol{q} \wedge \dot{\boldsymbol{q}} + \operatorname{grad} \mathrm{U}(q) ,$$

$$(2.12) \qquad \mathbf{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{\overline{\mathrm{K}}}{q^3} \boldsymbol{q} \wedge \dot{\boldsymbol{q}} + \operatorname{grad} \mathrm{U}(q) , \qquad \textit{where} \quad \overline{\mathrm{K}} = \mathrm{K} \frac{\mathrm{M}+m}{\mathrm{M}-m} ;$$

(ii) *with respect to* $\mathcal{R}_{\mathcal{T}\mathcal{B}}$, $|\mathbf{L}^*|$ *is a first integral of the motion of* $\mathrm{P} : d|\mathbf{L}^*|/\mathrm{d}t = 0$;

(iii) *the motions of* P *compatible with the forces* (2.11) *have trajectories in* $\mathcal{R}_{\mathcal{C}\mathcal{B}}$ *lying* (*each*) *on a fixed conic surface* (*in* $\mathcal{R}_{\mathcal{C}\mathcal{B}}$) *that has the vertex at* S, *the axis parallel with* $\hat{\omega}^A$ (*hence* $d\hat{\omega}^A/dt = \mathbf{0}$) *and semi-aperture* σ, *where*

$$\sigma = \arc\cos |\hat{\omega}^A \times \hat{q}| = \arc\cos \left\{ \frac{|K|}{\sqrt{K^2 + |\mathbf{L}^*|^2}} \right\} . \tag{2.13}$$

It easy to note that usual theory $\mathcal{C}$ is the special case of $\mathcal{C}^*$ obtained for $K = 0$, in this case we have $\sigma = \pi/2$, i.e. the conic surface reduces to a plane.

The next (additional) axiom PM—i.e. axiom A 6 in [2]—on the orbits of P in $\mathcal{R}_{\mathcal{C}\mathcal{B}}$ is reached by qualitative observations; it is a theorem in $\mathcal{T}$ (where $K = 0$), and is certainly weaker than e.g. the requirement that these orbits should be conic.

PLANE MOTIONS AXIOM (PM). *In* $\mathcal{R}_{\mathcal{C}\mathcal{B}}$ (*with the origin in* S) *the motion of* P *is plane.*

THEOREM 2. *In* $\mathcal{C}^* + \mathrm{PM}$

(i) P'*s trajectory in* $\mathcal{R}_{\mathcal{C}\mathcal{B}}$ *is a conic*;

(ii) *the central component of* $\mathfrak{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ (*i.e. parallel with* $\boldsymbol{q}$) *is quasi-Newtonian*:

$$\mathfrak{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{K}{q^3} \boldsymbol{q} \wedge \dot{\boldsymbol{q}} + \left(\frac{\gamma}{q^3} + \frac{K^2}{m^* q^4} \right) \boldsymbol{q} , \tag{2.14}$$

that is (2.11) *holds with the potential*

$$U(q) = -\frac{\gamma}{q} - \frac{K^2}{2m^*} \frac{1}{q^2} + \text{const} . \tag{2.15}$$

where K *and* γ *are independent real constants.*

For $K = 0$ we obtain Newton's law.

3. ON THE PRECESSION OF THE APSIDAL POINTS OF ORBITS IN $\mathcal{T}^*$

It is easy to check that the precession of the apsidal points for the orbits of planets is not describable in $\mathcal{C}^* + \mathrm{PM}$. In fact in this theory P's orbit in $\mathcal{R}_{\mathcal{C}\mathcal{B}}$ belongs to a cone and a plane that are both fixed in $\mathcal{R}_{\mathcal{C}\mathcal{B}}$ and are determined by the initial conditions $|\mathbf{L}^*|$, K, and γ. Therefore let us consider the more general theory $\mathcal{C}^*$.

We want a determination of the force function $\mathfrak{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$—cf. (2.11)—for which (i) P's motion (in $\mathcal{R}_{\mathcal{C}\mathcal{B}}$) is nearly plane (for $|K| \ll 1$), but not exactly plane [1], and (ii) the corresponding force function $\mathbf{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in any inertial fra-

(1) Otherwise no precession can arise, as was shown in $\mathcal{C}^* + \mathrm{PM}$.

me $\mathscr{R}_{\mathscr{I}}$ satisfies the hypothesis of linear dependence on masses, e.i. (2.12) with $\overline{\mathrm{K}} = h\,\mathrm{M}m$. Remembering the results obtained in $\mathscr{C}^* + \mathrm{PM}$, and in particular (2.14), the most natural choice for $\mathfrak{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ compatible with (i) and (ii) is

$$\mathfrak{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{\mathrm{K}}{q^3}\,\boldsymbol{q}\wedge\dot{\boldsymbol{q}} + \operatorname{grad}\mathscr{U}(q)\,, \tag{3.1}$$

with

$$\mathrm{K} = h\,\mathrm{M}m\,\frac{\mathrm{M}-m}{\mathrm{M}+m}\quad,\quad \mathscr{U}(q) = \mathrm{G}\,\frac{\mathrm{M}m}{q}\qquad (\gamma = -\,\mathrm{G}\,\mathrm{M}m)\,. \tag{3.2}$$

where G is Cavendish's gravitation constant and h is another universal constant. By (3.1), (2.12) becomes

$$\mathbf{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{h\mathrm{M}m}{q^3}\,\boldsymbol{q}\wedge\dot{\boldsymbol{q}} - \frac{\mathrm{G}\mathrm{M}m}{q^3}\,\boldsymbol{q}\,, \tag{3.3}$$

or

$$\mathbf{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{\mathrm{G}\,\mathrm{M}m}{q^3}\,(1 + \varepsilon\dot{\boldsymbol{q}}\wedge)\,\boldsymbol{q}\,,\qquad \varepsilon = -\frac{h}{\mathrm{G}}\,. \tag{3.4}$$

The latter formula appears easily comparable with the one proposed by Armellini in [1] as a correction of Newton's law

$$\mathbf{F}'(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{\mathrm{G}\mathrm{M}m}{q^3}\,(1 + \varepsilon\dot{q})\,\boldsymbol{q}\qquad (\dot{q} = |\dot{\boldsymbol{q}}|)\,. \tag{3.5}$$

Unlike (3.5), (3.4) admits a generalized energy integral in $\mathscr{R}_{\mathscr{C}\mathscr{B}}$.

Let us specialize the system $\mathscr{R}_{\mathscr{C}\mathscr{B}} = \{\mathrm{S}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ by assuming that $\boldsymbol{e}_3 = \hat{\omega}^{\mathrm{A}}$ cf. Theorem 1 (iii). In the system of spherical co-ordinates $\{r, \vartheta, \varphi\}$ (2) associated with $\mathscr{R}_{\mathscr{C}\mathscr{B}}$ we have:

$$\vartheta = \sigma = \text{const.}\,,\qquad \dot{\vartheta} = 0\,. \tag{3.6}$$

Let us determine the precession angle φ for the points (if any) where r attains a maximum $r_{\max}$ [a minimum $r_{\min}$]. This precession is a natural analogue of the precession of apsidal points for plane orbits.

Let us remember that in spherical co-ordinates $\boldsymbol{q} = r\hat{q}$, $\dot{\boldsymbol{q}} = v_q\,\hat{q} + v_\vartheta\,\hat{\vartheta} + v_\varphi\,\hat{\varphi} = \dot{r}\,\hat{q} + r\dot{\vartheta}\,\hat{\vartheta} + r\dot{\varphi}\sin\vartheta\,\hat{\varphi}$, while $\ddot{\boldsymbol{q}} = a_q\,\hat{q} + a_\vartheta\,\hat{\vartheta} + a_\varphi\,\hat{\varphi}$, where

$$a_q = \ddot{r} - r\dot{\vartheta}^2 - r\dot{\varphi}^2\sin^2\vartheta\,. \tag{3.7}$$

By (3.6)

$$a_q = \ddot{r} - r\dot{\varphi}^2\sin^2\sigma\,. \tag{3.8}$$

(2) $r = q$, ϑ is the colatitude, and φ is the longitude.

The vector

$$\mathfrak{L} = m\, \boldsymbol{q} \wedge \dot{\boldsymbol{q}} = \frac{m}{m^*} \mathbf{L}^* \tag{3.9}$$

is the *moment of momentum* in $\mathcal{R}_{\mathcal{C}\mathcal{B}}$ for the system $\{S , P\}$. Then

$$\mathfrak{L} = m\boldsymbol{q} \wedge (\dot{r}\hat{q} + r\dot{\vartheta}\hat{\vartheta} + r\dot{\varphi}\sin\vartheta\hat{\varphi}) = mr^2\,\dot{\varphi}\sin\sigma\,\hat{q}\wedge\hat{\varphi}\,, \tag{3.10}$$

so that

$$\dot{\varphi} = \frac{\mathcal{L}}{mr^2\sin\sigma} \qquad (\mathcal{L} = |\,\mathfrak{L}\,|)\,. \tag{3.11}$$

Let $r = r(\varphi)$ represent P's trajectory. Then

$$\dot{r} = \frac{dr}{d\varphi}\,\dot{\varphi} = \frac{dr}{d\varphi}\,\frac{\mathcal{L}}{mr^2\sin\sigma} = -\,\frac{\mathcal{L}}{m\sin\sigma}\,\frac{d}{d\varphi}\left(\frac{1}{r}\right),$$

whence

$$\ddot{r} = -\left(\frac{\mathcal{L}}{mr\sin\sigma}\right)^2 \frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right). \tag{3.12}$$

On the other hand, for the radial acceleration a_q, we have the kinematic relation (3.8) and the dynamic one:

$$m^*\,a_q = -\,\frac{GMm}{r^2} \qquad \text{(cf. (3.1–2))}. \tag{3.13}$$

For $u = r^{-1}$, by (3.11) to (3.13), (3.8) becomes

$$-\,\frac{GMmu^2}{m^*} = -\left(\frac{\mathcal{L}}{m\sin\sigma}\right)^2 u^2\,\frac{d^2 u}{d\varphi^2} - \left(\frac{\mathcal{L}}{m\sin\sigma}\right)^2 u^3\sin^2\sigma\,,$$

whence we have the equation

$$\frac{d^2 u}{d\varphi} + u\sin^2\sigma = \frac{G\,Mm^3\sin^2\sigma}{m^*\,\mathcal{L}^2} \tag{3.14}$$

in $u(\varphi)$, whose general solution is

$$\tilde{u}(\varphi) = \mathcal{A}_1\cos(\varphi\sin\sigma + \mathcal{A}_2)\,, \qquad (\mathcal{A}_1\,,\,\mathcal{A}_2 \in \mathbf{R})\,. \tag{3.15}$$

The extremal points $\overline{\varphi}$ for $\tilde{u}$ satisfy the equation

$$0 = \frac{d\tilde{u}}{d\varphi}(\overline{\varphi}) = -\,\mathcal{A}_1\sin\sigma\sin(\overline{\varphi}\sin\sigma + \mathcal{A}_2)\,. \tag{3.16}$$

Hence, setting

$$\varphi_n\sin\sigma + \mathcal{A}_2 = n\pi\,, \tag{3.17}$$

and assuming e.g. $\mathscr{A}_1 > 0$, φ_n is a maximum or minimum point for $\tilde{u}$ according to whether n is even or odd. Hence the dihedral angle of which the plane $(S, e_3, \hat{q})$ and the vector $\boldsymbol{q}$ (on the cone of semi-aperture σ) rotate between two passages through the same (generalized) apsidal point is $\breve{\varphi} = \varphi_{n+2} - \varphi_n$, that is

$$\breve{\varphi} = \frac{2\pi}{\sin\sigma}. \tag{3.18}$$

By (2.13) $\sigma = \arc\cos|\hat{\omega}^A \times \hat{q}| = \arc\cos\{|K|/\sqrt{K^2 + (L^*)^2}\}$, $L^* = |\mathbf{L}^*|$, whence $\sin\sigma = 1/\sqrt{1 + (K/L^*)^2}$ and

$$\breve{\varphi} = 2\pi\sqrt{1 + (K/L^*)^2}. \tag{3.19}$$

We have the representation

$$\breve{\varphi} = 2\pi\left[1 + \frac{1}{2}\left(\frac{K}{L^*}\right)^2\right] + 0\left[\left(\frac{K}{L^*}\right)^2\right] \qquad \left(\lim_{\xi\to 0}\frac{0(\xi)}{\xi} = 0\right). \tag{3.20}$$

Let us introduce the (*revolution*) *precession angle* $\Delta\varphi$:

$$\Delta\varphi = \breve{\varphi} - 2\pi, \tag{3.21}$$

whence, up to $0[(K/L^*)^2]$, we have

$$\Delta\varphi = \pi(K/L^*)^2. \tag{3.22}$$

By $(3.2)_1$ and $(3.9)_2$, (3.22) becomes

$$\Delta\varphi = \pi h^2 M^2 m^2 \left(\frac{M - m}{M}\right)^2 \frac{1}{\mathscr{L}^2}. \tag{3.23}$$

Thus we have obtained an expression for the precession angle very close to the well known one that is deduced by a suitable approximation procedure on the basis of General Relativity—cf. [3], § 98—:

$$\Delta\varphi_{GR} = \frac{6\pi G^2 M^2 m^2}{c^2} \cdot \frac{1}{\mathscr{L}^2} \qquad (c\text{: speed of light in vacuo}). \tag{3.24}$$

It is clear that for $M \gg m$, $(M - m)^2 M^{-2} \approx 1$. By identifying the universal constant h with $\sqrt{6}\,G/c$, (3.23) becomes

$$\Delta\varphi = \frac{6\pi^2 G^2 M^2 m^2}{c^2}\left(\frac{M - m}{M}\right)^2 \frac{1}{\mathscr{L}^2}, \tag{3.25}$$

which describes the precession of Mercury's perihelion as well as the analogues (difficult to observe) for the other planets, with a precision comparable the one of the relativistic formula (3.24) [(3)].

References

[1] G. Armellini (1937) – *I problemi fondamentali della Cosmogonia e la legge di Newton*, « Rend. Acc. Naz. dei Lincei », serie VI, vol. XXVI, p. 209.

[2] F. Cardin (1982) — *A generalized theory of classical mechanics for the two body problem*, to appear on « Rend. Sem. Mat. Univ. di Padova », vol. 69.

[3] L. Landau and E. Lifchitz (1970) – *Théorie Des Champs*, Editions MIR Moscou.

(3) The values of $\left(\frac{M-m}{M}\right)^{-2}$ for Earth and Mercury are $1 + 6.024 \cdot 10^{-6}$, and $1 + 3.37 \cdot 10^{-7}$ respectively.