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On extrapolation spaces

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Analisi matematica. — *On extrapolation spaces.* Nota di GIUSEPPE DA PRATO (*) e PIERRE GRISVARD, presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si definisce un nuovo tipo di spazi a partire da un dato spazio di Banach X e da un operatore lineare A in X. Tali spazi si possono pensare come spazi di interpolazione $D_A(\vartheta)$ con ϑ negativo.

1. EXTRAPOLATION SPACES

Let X be a Banach space and let $A : D(A) \subset X \to X$ be a closed operator densely defined in E. We assume:

(1) *The resolvent set of* A, $\rho(A)$, *contains* $[0 + \infty[$ *and there exists* $M_A > 0$ *such that*

$$|(\lambda - A)^{-1}|_{X \to X} \leq M_A/\lambda \qquad \forall \lambda > 0 .$$

We set

$$G_A = \{(x, y) \in X \times X ; x \in D(A), Ax = y\} \tag{2}$$

and denote by F the quotient space $F = (X \times X)/G_A$ and by $(x, y)^{\sim}$ the coset of (x, y).

The natural injection of X in F is defined by

$$J(x) = (0, x)^{\sim} \qquad \forall x \in X . \tag{3}$$

We can define an extension of A to J (X) by setting

$$\tilde{A}(0, x)^{\sim} = -(x, 0)^{\sim} \qquad \forall x \in X \tag{4}$$

PROPOSITION 1. *Under hypothesis* (1), $\rho(\tilde{A}) \supset]0, +\infty[$ *and*

$$|(\lambda - \tilde{A})^{-1}|_{F \to F} \leq M_A/\lambda \qquad \forall \lambda > 0 . \tag{5}$$

Moreover for any $\theta \in]0, 1[$ *we have*

$$D_{\tilde{A}}(\vartheta + 1) = J(D_A(\vartheta)) . \tag{6}$$

Here $D_A(\vartheta)$ represents the continuous interpolation space defined in [1].

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We set

$$D_{\tilde{A}}(\vartheta) = D_A(\vartheta - 1) \qquad \forall \vartheta \in]0,1[\tag{7}$$

and identify $D_{\tilde{A}}(\vartheta + 1)$ with $D_A(\vartheta)$.

Consider now another linear operator $B : D(B) = D(A) \to X$. We assume:

$$\begin{cases} i) & A^{-1} B \quad \textit{is continuous in } X; \\ ii) & \rho(B) \supset \mathbf{R}_+ \quad , \quad |(\lambda - B)^{-1}| \leq M_B/\lambda , \qquad \forall \lambda > 0; \\ iii) & B^{-1} A \quad \textit{is continuous in } X; \end{cases} \tag{8}$$

and set

$$\begin{cases} \tilde{B}J(x) = -(\overline{A^{-1}B}\, x, 0)^{\sim}; \\ D(\tilde{B}) = J(X); \end{cases} \tag{9}$$

where $\overline{A^{-1}B}$ is the closure of $A^{-1}B$.

PROPOSITION 2. *Under hypotheses* (1) *and* (8) *we have*

$$\begin{cases} i) & \rho(\tilde{B}) \supset \mathbf{R}_+ \quad , \quad |(\lambda - B)^{-1}| \leq M_B/\lambda \qquad \forall \lambda > 0; \\ ii) & D_{\tilde{B}}(\vartheta + 1) = D_{\tilde{A}}(\vartheta + 1) \qquad \forall \vartheta \in]0,1[. \end{cases} \tag{10}$$

2. EVOLUTION EQUATIONS

Let $\{B(t)\}_{t \in [0,T]}$ be a family of linear operators in X. We set $A = B(0)$ and assume:

(11)
- *i)* *For any* $t \in [0,T]$, $B(t)$ *generates an analytic semigroup* in X, $0 \in \rho(B(t))$.
- *ii)* $D(B(t)) = D(A)$ *and the norm of the graphs of* A *and* $B(t)$ *are equivalent.*
- *iii)* *The mapping* $B : [0,T] \to \mathscr{L}(D(A); X)$, $t \to B(t)$ *is continuous.*

We also set

$$H(t,s) = B(t)\,B^{-1}(s) \quad , \quad K_0(t,s) = B^{-1}(t)\,B(s) . \tag{12}$$

Concerning K_0 we assume:

(13)
- *i)* *For any* (t,s), $K_0(t,s)$ *is continuous in* X.
- *ii)* *If* K *denotes the closure of* K_0 *then the mapping*

$$K : [0,T] \times [0,T] \to \mathscr{L}(X)$$

is continuous.

THEOREM 1. *Assume that hypotheses* (11) *and* (13) *hold. Then for any* $x \in D_A(\vartheta)$ *and* $f \in C([0,T]; D_A(\vartheta - 1))$, $\vartheta \in]0,1[$ *there exists a unique strict solution of the Cauchy problem*

$$(14) \qquad \begin{cases} u'(t) = B(t)\,u(t) + f(t) \\ u(0) = x \end{cases}$$

with $u \in C^1([0,T]; D_A(\vartheta - 1)) \cap C([0,T]; D_A(\vartheta))$.

The proof uses an argument of [1] and relies essentially on the equality $D_{B(t)}(\vartheta + 1) = D_A(\vartheta + 1)$ which follows from Proposition 2. The details of the proofs as well as some applications will appear in a forthcoming paper.

REFERENCES

[1] G. DA PRATO and P. GRISVARD (1979) - *Equations d'évolution abstraites non linéaires de tipe parabolique.* « Annali di Matematica Pura e Applicata », Vol. CXX pp. 329-396.