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A remark on hyper-indecomposable groups

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Teoria dei gruppi. — *A remark on hyper-indecomposable groups* (*).

Nota di LADISLAV BICAN, presentata (**) dal Corrisp. I. BARSOTTI.

RIASSUNTO. — Un gruppo abeliano senza torsione ed indecomponibile è detto iperindecomponibile se tutti i sottogruppi propri del suo inviluppo iniettivo che lo contengono sono indecomponibili. In questo lavoro si caratterizza la classe dei gruppi iperindecomponibili per mezzo di loro proprietà locali. I gruppi iperindecomponibili omogenei sono caratterizzati tramite la proprietà « factor-splitting ».

An indecomposable torsionfree abelian group is said to be hyper-indecomposable if all proper subgroups between its divisible hull and itself are indecomposable. The purpose of this brief note is to describe the class of hyper-indecomposable groups by local properties and to prove that the homogeneous groups from this class are characterized by the factor-splitting property (the existence of homogeneous and non-homogeneous hyper-indecomposable groups up to rank $2^{\aleph_0}$ is proved in [7]). For the sake of completeness we include the descriptions of hyper-indecomposable groups obtained by Benabdallah and Birtz in [1].

All the groups considered are abelian. The set of all integers is denoted by $\underline{Z}$, $\underline{N}$ is the set of all positive integers, $\underline{N}_0 = \underline{N} \cup \{0\}$, and $\underline{Z}_p$ is the group of all rationals with denominators prime to p. If G is a (mixed) group then the symbol $h_p^G(g)$ ($\tau^G(g)$, $\hat{\tau}^G(g)$ resp.) denotes the p-height (the characteristic, the type resp.) of the element g in the group G. The divisible hull of a group G is denoted by D(G); $G[p^\infty]$ is the subgroup of G consisting of all elements of infinite p-height. Other notation and terminology will be essentially that as is [8].

Recall some basic definitions. The elements $x_1, x_2, \cdots, x_n$ of a torsionfree group G are said to be p-independent in G if any relation $px = \sum_{i=1}^{n} a_i x_i$, $a_1, a_2, \cdots, a_n \in \underline{Z}$, $x \in G$, implies $p \mid a_i$, $i = 1, 2, \cdots, n$. If $\alpha_i = \{a_i^{(k)}\}_{k=1}^{\infty}$, $a_i^{(k)} \in \underline{Z}$, $0 \le a_i^{(k)} < p^k$, $a_i^{(k)} \equiv a_i^{(k+1)} \pmod{p^k}$, $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots$, are p-adic integers then $\sum_{i=1}^{n} \alpha_i x_i \equiv 0 \pmod{p^\infty}$ means that for every $k = 1, 2, \cdots$ it is $\sum_{i=1}^{n} \alpha_i^{(k)} x_i = p^k x^{(k)}$ for suitable $x^{(k)} \in G$. If the relation $\sum_{i=1}^{n} \alpha_i x_i \equiv 0 \pmod{p^\infty}$ always implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ then the elements $x_1, x_2, \cdots, x_n$ are said to be p^∞-independent. A subset $M \subseteq G$ is called p-independent

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(p^∞-independent) if each of its finite subsets is so. Every maximal p-independent (p^∞-independent) subset of G is a p-basis (p^∞-basis). We denote by p^∞-dim G the cardinality of any p^∞-basis of G. It is easy to see that every p-independent set is p^∞-independent and consequently independent (see [9]).

LEMMA. *Let g, h be independent elements of a torsionfree group* G *and let* $\langle g, h\rangle_p$ *be the p-pure closure of* $\langle g, h\rangle$ *in* G. *Then the following are equivalent*:

(*i*) g, h *are* p^∞*-independent*;

(*ii*) $\langle g, h\rangle_p / \langle g, h\rangle$ *is finite*;

(*iii*) $\langle g, h\rangle_p$ *contains p-independent elements* a, b.

Proof. (*i*) $\Longleftrightarrow$ (*ii*). See [9] Lemma 1.

(*ii*) $\Rightarrow$ (*iii*). Without loss of generality we can assume that $h_p^G(g) = h_p^G(h) = 0$. It is easy to see that then $\langle g, h\rangle_p / \langle g, h\rangle$ is a cyclic group of order p^k for some $k \in \underline{N}_0$. Now $p^k b = g + rh$ for suitable $b \in \langle g, h\rangle_p$, $r \in \underline{Z}$, and $a = g$, b are p-independent in G, for otherwise $pw = sa + tb$, $w \in G$, $s, t \in \underline{Z}$, gives $p^{k+1} w = (p^k s + t) g + trh$ and $p \mid t$, $\langle g, h\rangle_p / \langle g, h\rangle$ being of order p^k. Then $p \mid s$ owing to the hypothesis $h_p^G(g) = 0$.

(*iii*) $\Rightarrow$ (*ii*). Obviously, $\alpha g, \beta h \in \langle a, b\rangle$ and $p^r a, p^r b \in \langle g, h\rangle$ for suitable $\alpha, \beta, r \in \underline{N}$. If $x \in \langle g, h\rangle_p$ is arbitrary, $p^k x \in \langle g, h\rangle$, then $p^k \alpha\beta x \in \langle a, b\rangle$ and consequently $\alpha\beta x \in \langle a, b\rangle$, a, b being p-independent. Thus $p^r \alpha\beta x \in \langle g, h\rangle$, $\langle g, h\rangle_p / \langle g, h\rangle$ is bounded and hence finite.

A torsionfree group G is said to be hyper-indecomposable if all proper subgroups of D (G) containing G are indecomposable. D. W. Dubois [7] has called a torsionfree group G cohesive if G/S is divisible for every non-zero pure subgroup S of G.

If p is a prime, then a torsionfree group G of rank at least 2 is said to be a p-i-group (p-irational group) if for every pair (a, b) of independent elements of G and every $i \in \underline{N}$ there is $n_i \in \underline{N}_0$, $n_i < p^i$, such that $h_p^G(a + p^{\alpha-\beta} n_i b) \geq i + \alpha$, $\alpha = h_p^G(a)$, $\beta = h_p^G(b)$, and the p-adic number $\eta(a, b) = \lim_i p^{\alpha-\beta} n_i$ is not rational (see [1]).

THEOREM 1. *Let* G *be a reduced torsionfree group and* D *be its divisible hull. Then the following are equivalent*:

(*i*) *For every prime p with* $G \neq pG$ *it is* $G[p^\infty] = 0$ *and* $p^\infty - \dim G = 1$;

(*ii*) *For every prime p with* $G \neq pG$ *it is* $G[p^\infty] = 0$ *and* $|G/pG| = p$;

(*iii*) G *is a p-i-group for every prime p with* $G \neq pG$;

(*iv*) G *is cohesive*;

(*v*) $G + E = D$ *for every non-zero divisble subgroup* E *of* D;

(*vi*) G *is hyper-indecomposable*;

(*vii*) *For every prime* p, $G_p = G \otimes \underline{Z}_p$ *is either divisible or hyper-indecomposable*;

(*viii*) G_p *is reduced and* $D/G_p \cong Z(p^\infty)$ *for every prime* p *with* $G_p \neq D$;

(*ix*) G_p *is cohesive for every prime* p.

Proof. (*i*) $\Rightarrow$ (*ii*). Any two elements of G are p-dependent.

(*ii*) $\Rightarrow$ (*i*). If G contains two p^∞-independent elements g, h then $\langle g, g\rangle_p$ contains two p-independent elements by Lemma.

(*i*) $\Rightarrow$ (*iii*). Obviously, if $a, b \in G$ are independent then $\eta_p(a, b)$ is not rational if and only if $|\langle a, b\rangle_p / \langle a, b\rangle| = \infty$, which means that a, b are p^∞-dependent (by Lemma).

(*i*) $\Rightarrow$ (*iv*). Let $S \neq 0$ be pure in G, $g \in G \setminus S$ be arbitrary. If p is a prime with $G \neq pG$, choose an element $h \in S$ with $h_p^G(g) = h_p^G(h)$. Then the p–dependence of the elements g, h gives the existence of a p-adic integer $\alpha = \{a_k\}_{k=1}^\infty$ such that $g + \alpha h \equiv 0 \pmod{p^\infty}$. So $p^k x_k = g + a_k h$ for suitable $x_k \in G$; hence $p^k(x_k + S) = g + S$, $k \in \underline{N}$, and G/S is divisible.

(*iv*) $\Rightarrow$ (*v*). The subgroup $S = E \cap G$ is pure in G and so $(G + E)/E \cong G/S$ is divisible. Thus $G + E$ is divisible.

(*v*) $\Rightarrow$ (*vi*). Let a subgroup H, $G \subseteq H \subsetneqq D$, be decomposable, $H = H_1 \oplus H_2$, and let p be a prime such that H is not p-divisible. Then one of H_1, H_2, say H_1, is not p-divisible and $G + D(H_2) \neq D$ contradicts the hypothesis.

(*vi*) $\Rightarrow$ (*vii*). Obvious, since $G_p \subseteq D_p = D$.

(*vii*) $\Rightarrow$ (*viii*). If $G_p \neq D$ then G_p is clearly reduced and if E is a rank one pure subgroup of D then $G_p + E = D$ and $D/G_p \cong E/G_p \cap E \cong Z(p^\infty)$.

(*viii*) $\Rightarrow$ (*ix*). If G_p is not divisible and S is pure in G_p, $E = D(S)$, then $(G_p + E)/G_p \cong E/E \cap G_p$ is divisible. Hence $G_p + E = D$ and $G_p/S = G_p/G_p \cap E \cong D/E$ is divisible.

(*ix*) $\Rightarrow$ (*i*). If $G \neq pG$ then G_p is reduced and so $G[p^\infty] = 0$. Let g, h be independent elements of G. Then $G_p/\langle h\rangle_*$ (the pure closure in G_p) is divisible and h can be chosen such that $h_p^G(g) = h_p^G(h)$. Hence for each $k \in \underline{N}$ there is $a_k \in \underline{N}$ and $x_k \in G_p$ with $p^k x_k = g + a_k h$. Now it is easy to see that $x_k \in G$ and $\alpha = \{a_k\}_{k=1}^\infty$ is a p-adic integer; therefore $g + \alpha h \equiv 0 \pmod{p^\infty}$ proves the p^∞-dependence of g, h in G.

Remark. The equivalence of (*ii*) and (*iv*) has been proved in [7] while the equivalence of (*iii*), (*iv*), (*v*) and (*vi*) has been proved in [1]. It should be noted that by a slight modification of some examples in [8] the indecomposable non-cohesive $\underline{Z}_p$-modules can be constructed.

A sequence $g_0, g_1, \cdots$ of elements of a mixed group G is said to be a p-sequence of g_0 if $pg_{i+1} = g_i$, $i = 0, 1, \cdots$. If G is a mixed group with the torsion part T such that G/T is divisible then G splits if, and only if, every element $g \in G \setminus T$ has a non-zero multiple mg which has a p-sequence in G for each prime p (see [3] and [2]).

Recall that a torsionfree group G is said to be factor-splitting if all of its factor-groups split. If G is factor-splitting then every pure subgroup of G is factor-splitting, and if G is of rank two then it is factor-splitting if, and only if, for any two independent elements $g, h \in G$ it is $(\langle g \rangle_* \oplus \langle h \rangle_*) \otimes \underline{Z}_p = G \otimes \underline{Z}_p$ for almost all primes p with $h_p^G(g) \neq h_p^G(h)$ (see [6] and [4]).

Theorem 2. *A hyper-indecomposable torsionfree group* G *is homogeneous if, and only if, it is factor-splitting.*

Proof. Suppose that G is a non-divisible homogeneous hyper-indecomposable group. Let $H \neq 0$ be an arbitrary subgroup of G and $S = \langle H \rangle_*$ be the pure closure of H in G. By Theorem 1 (*iv*) G/S is divisible, so that with respect to Corollary 2 of [3] it suffices to show that every element $g' \in G \setminus S$ has a non-zero multiple g such that the element $g + H$ has a p-sequence in G/H for each prime p. If $0 \neq h \in H$ is arbitrary then every element $g' \in G \setminus S$ has a non-zero multiple g with $\tau^G(g) \geq \tau^G(h)$, G being homogeneous. If G is p-divisible then $g + H$ has obviously a p-sequence in G/H. If G is not p-divisible then the elements g, h are p^∞-dependent by Theorem 1 (*i*) so that there is a p-adic integer $\alpha = \{a_k\}_{k=1}^\infty$ such that $g + \alpha h \equiv 0 \pmod{p^\infty}$. Consequently, for each $k \in \underline{N}$ it is $p^k x_k = g + a_k h$ for suitable $x_k \in G$ and if $a_{k+1} = a_k + p^k t_k$, $t_k \in \underline{N}_0$, then $p^{k+1} x_{k+1} = g + a_{k+1} h = p^k (x_k + t_k h)$ yields $p x_{k+1} = x_k + t_k h$ and $x_0 + H = g + H$, $x_1 + H, \cdots$ is the desired p-sequence.

Suppose now that G is factor-splitting and let $g, h \in G$ be elements of different types. By Lemma 2 of [6] the pure subgroup $S = \langle g, h \rangle_*$ of G is factor-splitting and consequently, by Theorem 1 of [4], for almost all primes p with $h_p^G(g) \neq h_p^G(h)$ it is $(\langle g \rangle_* \oplus \langle h \rangle_*)_p = S_p$. This contradicts Theorem 1 (*ix*) since any pure subgroup of a cohesive group is obviously cohesive.

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