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Daniele Mundici

L-embedding, Amalgamation and L-elementary equivalence

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Logica matematica. — L-*embedding*, *Amalgamation and* L-*elementary equivalence*. Nota di DANIELE MUNDICI, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Ogni logica L genera canonicamente la L-equivalenza $\equiv_L$ e la L-immersione $\xrightarrow[L]{}$ proprio come la logica del primo ordine genera l'equivalenza elementare $\equiv$ e l'immersione elementare $\lesssim$. Astraendo da L, è interessante studiare in sè relazioni d'equivalenza e di immersione generali tra strutture. Mostriamo che esiste una corrispondenza biunivoca tra relazioni d'equivalenza con la proprietà di Robinson e relazioni d'immersione con la proprietà di Amalgamazione Forte (AP^+). Caratterizziamo algebricamente quelle relazioni di immersione che si possono scrivere come $\xrightarrow[L]{}$ per L una logica. Mostriamo che $\lesssim$ è generata esclusivamente dalla logica del primo ordine.

See [1]—[4] for the necessary background in Abstract Model Theory. Given a logic L one obtains an equivalence relation $\equiv_L$ on the class of all structures (by saying the $\mathfrak{A} \equiv_L \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of L) and an embedding relation $\xrightarrow[L]{}$ (by saying that $\mathfrak{A} \xrightarrow[L]{} \mathfrak{B}$ if the type $\tau_{\mathfrak{B}}$ of $\mathfrak{B}$ contains $\tau_{\mathfrak{A}}$ and for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$ we have that $\mathfrak{A}_A \equiv_L \mathfrak{B}^+$, where $\mathfrak{A}_A$ is the diagram expansion of $\mathfrak{A}$). The study of equivalence relations *in se* was initiated in [9] and pursued further in [5]. Recall that an (*abstract*) *equivalence relation* in this framework means an equivalence relation between structures of the same type, which is preserved under reduct, renaming and isomorphism. If $\sim$ is an abstract equivalence relation and $\sim = \equiv_L$ for some logic L, then automatically $\sim$ is a refinement of (first-order) elementary equivalence, $\equiv$.

The study of abstract embedding relations was initiated by the present author in 1979 (see [6] and [7]), using the following.

DEFINITION. A binary relation $\rightarrow$ on the class of all structures is an (*abstract*) *embedding relation* if $\rightarrow$ satisfies the following axioms:

(*a*) : $\mathfrak{A} \rightarrow \mathfrak{B}$ implies $\tau_{\mathfrak{A}} \subseteq \tau_{\mathfrak{B}}$; $\mathfrak{A} \rightarrow \mathfrak{B}$ iff $\mathfrak{A} \rightarrow \mathfrak{B} \restriction \tau_{\mathfrak{A}}$;

(*b*) : $\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{A} \rightarrow \mathfrak{B}$;

(*c*) : $\mathfrak{A} \rightarrow \mathfrak{B}$ implies $\mathfrak{A}^\rho \rightarrow \mathfrak{B}^\rho$ for each renaming ρ ;

(*d*) : $\mathfrak{A} \rightarrow \mathfrak{B}$ implies $\mathfrak{A} \restriction \tau \rightarrow \mathfrak{B} \restriction \tau$;

(*e*) : $\mathfrak{A} \rightarrow \mathfrak{B}$ and $\mathfrak{B} \rightarrow \mathfrak{M}$ implies $\mathfrak{A} \rightarrow \mathfrak{M}$.

We say that $\rightarrow$ has the *Strong Amalgamation Property* (AP^+) iff whenever $\mathfrak{A} \leftarrow \mathfrak{N} \rightarrow \mathfrak{B}$ and $\tau_{\mathfrak{A}} \cap \tau_{\mathfrak{B}} = \tau_{\mathfrak{N}}$, then for some $\mathfrak{M}$ we also have $\mathfrak{A} \rightarrow \mathfrak{M} \leftarrow \mathfrak{B}$.

(*) Nella seduta del 25 giugno 1982.

The definition of *Amalgamation Property* (AP) is the same except for the stronger requirement $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}} = \tau_{\mathfrak{M}}$. Every embedding relation $\rightarrow$ generates an equivalence relation $\sim\, = \,\rightarrow^*$ by saying that $\mathfrak{A} \sim \mathfrak{B}$ iff $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$ and $\mathfrak{A}$ and $\mathfrak{B}$ are connected by a finite path of arrows (e.g., $\mathfrak{A} \rightarrow \mathfrak{B}' \leftarrow \mathfrak{B}'' \rightarrow \mathfrak{B}''' \leftarrow \mathfrak{B}$) between structures of the same type. Conversely, every equivalence relation $\sim$ generates an embedding relation $\rightarrow\, = \,\sim^*$ by saying that $\mathfrak{A} \rightarrow \mathfrak{B}$ iff $\tau_{\mathfrak{B}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}_A \sim \mathfrak{B}^+$ for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$. Clearly, for any logic L, $\xrightarrow[L]{}$ is an embedding relation in the above general sense. All the above notions are naturally relativized to arbitrary classes X of structures (see [7]). Then we have:

THEOREM 1. *Let* L *be a compact logic where* $|\mathrm{Stc}_L\, \tau|$ *exists for each type* τ; *then the following are equivalent*:

(*i*) $\equiv_L\, = \,\equiv$ *upon restriction to the class of countable structures*;

(*ii*) $\xrightarrow[L]{}$ *has* AP+ *on the class of all countable structures of finite type*.

The proof is in [7] and uses the algebraic characterization of $\equiv$ on the countable structures with finite type given in [8]. Turning now to equivalence and embedding relations defined on the class of *all* structures, we say that an embedding $\rightarrow$ is *involutive* if $\rightarrow\, = \,\rightarrow^{**}$. Then we have (see [7] for a proof):

THEOREM 2. *Let* E *be a non empty class of embedding relations with* AP+ *such that, for some fixed equivalence relation* $\sim$, *we have that* $\rightarrow$ *is in* E *if* $\rightarrow^* = \sim$. *Then* E *possesses exactly one involutive element.*

Following [5]—[8] we say that an equivalence relation $\sim$ has the *Robinson property* iff whenever $\mathfrak{A} \restriction \tau_{\mathfrak{A}} \cap \tau_{\mathfrak{B}} \sim \mathfrak{B} \restriction \tau_{\mathfrak{A}} \cap \tau_{\mathfrak{B}}$, there exists $\mathfrak{M}$ such that $\mathfrak{M} \restriction \tau_{\mathfrak{A}} \sim \mathfrak{A}$ and $\mathfrak{M} \restriction \tau_{\mathfrak{B}} \sim \mathfrak{B}$. Let R be the collection of all equivalence relations with the Robinson property; let A be the collection of all involutive embedding relations with AP+:

THEOREM 3. *Over* $A \cup R$ *the function* * *maps* A *one-one onto* R *and maps* R *one-one onto* A; *furthermore*, ** *is the identity function on* $A \cup R$.

For a proof see [7]; we thus see that there is a duality between embeddings in A and equivalence relations in R. But then we can apply the other duality theorem, proved in [5], between compact logics with interpolation and equivalence relations in R, extending it to embeddings in A, as follows (proof in [7]):

THEOREM 4. *For an embedding relation* $\rightarrow$, *the following are equivalent*:

(*i*) $\rightarrow\, = \,\xrightarrow[L]{}$ *for a unique logic* L *such that* $|\mathrm{Stc}_L\, \tau|$ *always exists and each sentence of* L *has a finite type*; *in addition* L *is compact and satisfies interpolation*;

(*ii*) $\rightarrow$ *is involutive, has* AP+ *and generates (via* *) *an equivalence relation* $\sim$ *finer than* $\equiv$, *bounded and separable.*

(Recall that, as in [5], $\sim$ is *bounded* if for each type τ the collection of equivalence classes of structures of type τ has a cardinality; $\sim$ is *separable* iff whenever not-$\mathfrak{A} \sim \mathfrak{B}$ then there is a quantifier Q such that $\mathfrak{A} \not\equiv_{L(Q)} \mathfrak{B}$ and $\equiv_{L(Q)}$ is coarser than $\sim$). As a consequence we have the following:

THEOREM 5. *First-order logic* $L_{\omega\omega}$ *is the only logic* $L = L(Q^i)_{i \in I}$ *such that* $\xrightarrow[L]{} = \xrightarrow[L_{\omega\omega}]{}$.

References

[1] BARWISE K. J. (1974) – *Axioms for abstract model theory*, « Ann. Math. Logic. », 7, 221–265.

[2] FEFERMAN S. (1974) – *Two notes on abstract model theory* I, « Fund. Math. », *82*, 153–175.

[3] FLUM J. (1976) – *First-order logic and its extensions*, « Lecture Notes in Math. », Springer-Verlag, Berlin, vol. 499, 248–310.

[4] MAKOWSKY J. A. SHELAH S. and STAVI J. (1976) – *Δ-logics and generalized quantifiers*, « Ann. Math. Logic », *10*, 155–192.

[5] MUNDICI D. (1982) – *Duality between logics and equivalence relations*, « Transactions AMS », *270*, 111–129.

[6] MUNDICI D. (1983) – *Compactness* = JEP *in any logic*, « Fund. Math. », *116*, to appear.

[7] MUNDICI D. (1981–1982) – *Lectures on Abstract Model Theory* I, Preprint, n° 6, « Math. Inst. Univ. of Florence ».

[8] MUNDICI D. (1981) – *An algebraic result about soft model theoretical equivalence relations with an application to Friedman's fourth problem*, « Journal of Symb. Logic », *46*, 523–530.

[9] NADEL M. (1980) – *An arbitrary equivalence relation as elementary equivalence in an abstract logic*, « Zeit. math. Logik. », *26*, 103–109.