ATTI ACCADEMIA NAZIONALE DEI LINCEI

CLASSE SCIENZE FISICHE MATEMATICHE NATURALI

RENDICONTI

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Some results about compact logics

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **72** (1982), n.6, p. 308–311.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1982_8_72_6_308_0>

Logica matematica. — *Some results about compact logics.* Nota di PAOLO LIPPARINI, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Nell'ambito della Teoria dei Modelli Astratta è possibile dimostrare che una logica compatta L è univocamente determinata dalla sua relazione di L-elementare equivalenza (Teorema 1). Si enunciano poi alcuni risultati sulle logiche massime correlate a certe relazioni di equivalenza e sulle logiche compatte generate da qualche sistema di Fraissé-Ehrenfeucht.

The aim of this note is to announce some results of the author in Soft Model Theory: their proofs will appear in [3].

Soft (or Abstract) Model Theory studies the extensions of first-order logic ($L_{\omega\omega}$): in particular two kinds of problems are taken into account: i) the research for extensions of first-order logic having some of its good model-theoretical properties (for example satisfying compactness, Craig interpolation theorem, Löwenheim-Skolem theorems and so on); and ii) the characterization of existing logics as maximal with respect to certain properties.

For example, the first non trivial theorem of Abstract Model Theory, proved by Lindström in 1966 [2], says that $L_{\omega\omega}$ is the only compact logic satisfying the downward Löwenheim-Skolem theorem, so that it is maximal among those having these properties. Naturally, this gives negative answers to problem i); nevertheless, by a result of Shelah [11], compact proper extensions of $L_{\omega\omega}$ do exist.

These are only examples of the nature of Soft Model Theory: many other interesting results and applications can be found in the literature (see [1]). In particular some logical concepts are given a purely algebraic characterization: in [6] and [4] it is proved that: i) compactness is equivalent to the joint embedding property, and also to the amalgamation property for L-elementary embeddings; ii) compactness+Craig interpolation is equivalent to the Robinson property. (In passing we remark that the Robinson property has many useful consequences, see [7], [8], [9]).

The strength of these characterizations can be seen from the following example: Lindström proved that $L_{\omega\omega}$ is the only compact logic L generating $\equiv$ as L-elementary equivalence. But any logic generating $\equiv$ must satisfy the joint embedding property in view of i) above. So, the quoted proposition can be improved to: $L_{\omega\omega}$ is the only logic generating $\equiv$.

(*) Nella seduta del 25 giugno 1982.

In [7] similar theorems results are obtained for compact logics satisfying Craig in place of $L_{\omega\omega}$; but perhaps these results can be applied only to $L_{\omega\omega}$, since, as far as we know, Friedman's fourth problem is still open (the research for compact proper extensions of $L_{\omega\omega}$ satisfying Craig). Nevertheless our Theorem 1 says that the Craig assumption is unnecessary, and this makes the result 'non-empty'. For example, a concrete consequence is that there is no logic $L \neq L_{\omega\omega}(Q^{cf\omega})$ such that $\equiv_L = \equiv_{L_{\omega\omega}(Q^{cf\omega})}$.

The fact that algebraic properties of equivalence relations can be useful for the study of logics has led to the following problem: characterize those equivalence relations which are generated by some logic (see [7], [10]). A rather complete answer for Robinson logics is given in [9]; in proposition 1 we extend some of the results of the latter paper to any countably generated logic.

Theorems 1 and 3 support the author's belief that, with few more technical efforts, many theorems about compact logics satisfying Craig can be proved assuming only compactness.

All notations and definitions are taken from [5], [9], [7]; we use the definition of logic of this last paper.

My thanks go to P. Mangani, my Research Director, and to D. Mundici, for his valuable help.

THEOREM 1. *If* L, M *are logics such that* $\mathrm{Stn}_L(\tau)$ *is a set for any* τ, $\equiv_M$ *is coarser than* $\equiv_L$, *and* L *is compact, then* $M \leq L$. *Moreover, if* $\equiv_M = \equiv_L$, *then* $M = L$.

DEFINITION 1. Let $\sim$ be any equivalence relation on the class of all structures. We say that a logic L *respects* $\sim$ iff whenever $\mathfrak{A} \sim \mathfrak{B}$ are two models of type τ, and $\varphi \in \mathrm{Stn}_L$ has type $\tau \cup \{c_1, \cdots, c_n\}$, if we add to $\mathfrak{A}$ and $\mathfrak{B}$ a new relation R defined by:

$$R(c_1, \cdots, c_n) \Longleftrightarrow \varphi,$$

then the two expanded models are also $\sim$-equivalent.

THEOREM 2. *Let* $\sim$ *be an equivalence relation finer than* $\equiv$ *and closed under renaming and reduct. If* L *and* M *respect* $\sim$, *then* $L \cup M$ *respects* $\sim$. *Thus, there exists the largest logic respecting* $\sim$. *Moreover, if* $L = L_{\omega\omega}(Q_i)_{j\in I}$, *then there exists the largest logic* M *of the form* $M = L_{\omega\omega}(Q_j)_{j\in J}$ *such that* $\equiv_L = \equiv_M$.

It can be proved that if $\sim$ has Robinson, then L respects $\sim$ if $\equiv_L$ is coarser than $\sim$, so Theorem 2 generalizes parts of Theorem 5.5 of [7] (see also Remarks 6.10).

DEFINITION 2. A FE-system satisfies the *expansion property* if, whenever $\mathfrak{A} \simeq_\tau^{m+n} \mathfrak{B}$, $\tau' = \tau \cup \{\bar{c}\}$, U is a union of classes of $\simeq_{\tau'}^n$, then $(\mathfrak{A}, R) \simeq_{\tau\cup\{R\}} (\mathfrak{B}, R)$, where R is defined in $\mathfrak{A}$ by

$$R\bar{a} \Longleftrightarrow (\mathscr{A}, \bar{a}) \in U,$$

and similarly for $\mathfrak{B}$.

PROPOSITION 1. *For any* $\sim$, *there is a* FE-*system satisfying expansion and generating* $\sim$ *if there is a countably generated logic* L *with* $\equiv_L = \sim$.

THEOREM 3. *For any* $\sim$, *the following are equivalent*:

i) $\sim = \equiv_L$ for some countably generated compact logic;

ii) $\sim = \equiv_L$ for exactly one logic L, which is compact and countably generated;

iii) $\sim$ has the joint embedding property and is generated by some FE-system satisfying expansion;

iv) $\sim$ has the joint embedding property and is generated by exactly one FE-system, which satisfies expansion.

ADDED IN PROOF (DECEMBER 1982). Other results are:

i) $L \cup M$ is $(\lambda, \mu)_\nu$-compact iff $S_\mu(\lambda)$ is not ν-atomically characterizable by $\equiv_L \cap \equiv_M$ (cf. [3]; $(\lambda, \mu)_\nu$-*compactness* is defined as $(\lambda, \mu)^*$-compactness in [4], but requiring $|\tau(\Sigma \cup \Sigma_1)| < \nu$).

ii) If $\equiv_L \subseteq \equiv_M$, $OC(M) \leq \nu$, L is (λ, ω)-compact and $|Stn_L(\tau)| \leq \lambda$ for $|\tau| < \nu$, then $M \leq L$.

iii) If $OC(L) \leq \lambda$, and $\equiv_L$ satisfies the Robinson property ([8]) with the restriction that $|\tau| \leq \lambda$ and $|\tau' \setminus \tau|, |\tau'' \setminus \tau| < \nu$, then L is $(\lambda, \omega)_\nu$-compact.

iv) If L is compact and generated by a set of quantifiers, then L satisfies Beth's Definability Theorem if $\equiv_L$ has the definability property ([3]).

v) If D is an ultrafilter, α an ordinal, and $\mathfrak{A} \sim \mathfrak{B}$ iff $\prod_D^\beta \mathfrak{A} \cong \prod_D^\gamma \mathfrak{B}$ for some $\beta, \gamma < \omega^\alpha$ (where $\prod_D {}^0\mathfrak{A} = \mathfrak{A}$, and $\prod_D^\beta \mathfrak{A} = \bigcup_{\gamma \in \beta} \prod_D (\prod_D^\gamma \mathfrak{A})$), then $\sim$ is a Robinson equivalence relation. If ω is the only measurable cardinal and $\sim = \equiv_L$, then $\sim = \cong$.

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