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Global existence for a Riccati equation arising in a boundary control problem for distributed parameters

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Analisi matematica. — *Global existence for a Riccati equation arising in a boundary control problem for distributed parameters.* Nota di FRANCO FLANDOLI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si prova l'esistenza globale della soluzione di una equazione di Riccati collegata alla sintesi di un problema di controllo ottimale. Il problema considerato rappresenta la versione astratta di alcuni problemi governati da equazioni paraboliche con il controllo sulla frontiera.

1. INTRODUCTION

Let U and H be the two real Hilbert spaces. Let A be a densely defined closed linear operator on H, with domain D (A). We assume that

(1) A is the infinitesimal generator of an analytic semigroup e^{tA}.

Let $D_A(\alpha, p)$ be the real interpolation spaces defined in [1] for $a \in (0,1)$, $p \in [1, +\infty]$. Let B be a linear bounded operator from U to H, such that:

(2) there exists $\alpha \in (0,1)$ such that $B \in \mathscr{L}(U, D_A(\alpha, \infty)$ (1).

We study the Riccati equation

$$(3) \quad \begin{cases} P'(t) = A^\star P(t) + P(t) A + M - P(t) ABN^{-1} B^\star A^\star P(t), & t \in [0,T] \\ P(0) = P_0, \end{cases}$$

where

$$(4) \qquad M \in \Sigma^+(H)\ ^{(2)} \quad , \quad N \in \Sigma^+(U) \quad , \quad N \geq \nu > 0 \quad , \quad P_0 \in \Sigma^+(H),$$

and we seek a solution $P(t) \in \Sigma^+(H)$. We note that the composition AB is not well defined; the meaning of (3) will be clear in section 2.

(*) Nella seduta del 24 aprile 1982.

(1) If X, Y are Banach spaces, $\mathscr{L}(X, Y)$ denotes the space of linear operators from X to Y, and $\mathscr{L}(X) = \mathscr{L}(X, X)$.

(2) $\Sigma^+(H) = \{T \in \mathscr{L}(H) \mid T = T^\star, \langle Tx, x\rangle_H \geq 0 \quad \forall x \in H\}$.

Equation (3) is connected (see [2]) with the synthesis of the following optimal control problem:

$$(5)\quad \begin{cases} \text{minimize } J(u) = \int_0^T \{ \langle My(t), y(t)\rangle_H + \langle Nu(t), u(t)\rangle_U \}\, dt \\ \text{over all controls } u \in L^2(0, T; U)^{(3)}, \end{cases}$$

where the state $y(t)$ is defined for a.e. $t \in [0, T]$ by

$$(6)\quad y(t) = e^{tA} x + \int_0^t A e^{(t-s)A} B u(s)\, ds,$$

and $x \in H$.

We note that (6) defines a state $y \in L^2(0, T; H)$.

It can be shown that (6) is a " mild " solution for a class of parabolic equations with " boundary " control (see [2], [3], [4]).

If we set $Q(t) = P(t) A$ and $Q_0 = P_0 A$ in equation (3), we obtain

$$(7)\quad \begin{cases} Q'(t) = A^\star Q(t) + Q(t) A + (M - Q(t) BN^{-1} B^\star A^\star Q(t) A, \\ Q(0) = Q_0; \end{cases}$$

using the variation of constant device we may write (7) in the integral form:

$$(8)\quad Q(t)\, x = e^{tA^\star} Q e^{tA} x + \int_0^t e^{(t-s)A^\star} (M - Q(s) BN^{-1} (Q(s) B)^\star) A e^{(t-s)A} x ds.$$

2. LOCAL EXISTENCE

The interpolation property of $D_A(\alpha, \infty)$ implies that

$$(9)\quad \text{there exists } c > 0 \text{ such that } \quad \| A e^{tA} x \|_H \le c \| x \|_{D_A(\alpha,\infty)}$$

$\forall x \in D_A(\alpha, \infty)$, $\forall t > 0$. Therefore we study equation (8) in the space $C_F([0, T]; \mathscr{L}(D_A(\alpha, \infty); H))$ of applications $U(\cdot)$ from $[0, T]$ to $\mathscr{L}(D_A(\alpha, \infty); H)$ such that $\forall x \in D_A(\alpha, \infty)$ the function $t \to U(t) x$ is continuous from $[0, T]$ to H. If $x \in D_A(\alpha, \infty)$, $Q \in C_F([0, T]; \mathscr{L}(D_A(\alpha, \infty); H))$, and $Q_0 \in \mathscr{L}(D_A(\alpha, \infty); H)$, then the integral in (8) is summable and

$$(10)\quad \gamma(Q)(t) = e^{tA^\star} Q_0 e^{tA} + \int_0^t e^{(t-s)A^\star} (M - Q(s) BN^{-1} (Q(s) B)^\star A e^{(t-s)A}\, ds$$

belongs to $C_F([0, T]; \mathscr{L}(D_A(\alpha, \infty); H))$.

(3) $L^2(0, T; U) = \left\{ f : [0, T] \to U, \text{ Bochner measurable}, \int_0^T \| f(t) \|_U^2\, dt < +\infty \right\}$.

Using a contraction principle in $C_F([0,T]; \mathscr{L}(D_A(\alpha,\infty); H))$ and some techniques as in [4] and [5], we can prove

(11) PROPOSITION. *Suppose* $Q_0 \in \mathscr{L}(D_A(\alpha,\infty); H)$ *and suppose that* (1), (2) *and* (4) *hold. Then for a suitable* $\tau > 0$ *there exists a unique solution* $Q \in C_F([0,\tau]; \mathscr{L}(D_A(\alpha,\infty); H))$ *of* (8) *(that is,* Q *verifies* (8) $\forall x \in D_A(\alpha,\infty)$*). Moreover there exists the unique maximal solution of* (8). *Furthermore if the "a priori" estimation* $\| Q(t) \|_{\mathscr{L}(D_A(\alpha,\infty);H)} \le c \quad \forall t$ *holds for the maximal solution, then there exists the global solution.*

Now we return to equation (3).

Let $P_0 \in \Sigma^+(H)$ such that there exists $Q_0 \in \mathscr{L}(D_A(\alpha,\infty); H)$ verifying $P_0 Ax = Q_0 x \ \forall x \in D(A)$. For this choice of Q_0 we have a solution Q of (8) over an interval $[0, T_1]$.

We set

$$(12) \qquad P(t)x = e^{tA^\star} P_0 e^{tA} x + \int_0^t e^{(t-s)A^\star} (M - Q(s) BN^{-1}(Q(s) B)^\star) e^{(t-s)A} x \, ds$$

$\forall t \in [0, T_1]$, $\forall x \in H$. $P \in C_F([0,T_1]; \Sigma^+(H))$, and $Q(t)x = P(t)Ax$ $\forall x \in D(A)$. We can prove, using some regularity results by [5]

(13) PROPOSITION. $\forall x, y \in D(A)$, $\forall t \in (0, T_1]$, *we have* $P(t)x \in D(A)^\star$, $t \to \langle P(t)x, y\rangle_H$ *differentiable, and*

$$\begin{cases} \langle P'(t)x, y\rangle_H = \langle Ax, P(t)y\rangle_H + \langle P(t)x, Ay\rangle_H + \langle Mx, y\rangle_H - \\ \qquad - \langle B^\star A^\star P(t)x, B^\star A^\star P(t)y\rangle_U \\ P(0) = P_0 . \end{cases}$$

Introducing the value function $J(t,x)$ $\forall t \in [0,T]$, $\forall x \in H$,

$$(14) \qquad J(t,x) = \inf \left\{ \int_t^T \{\langle My(\sigma), y(\sigma)\rangle_H + \langle Nu(\sigma), u(\sigma)\rangle_U\} \, d\sigma , \right.$$
$$\left. u \in L^2(t, T; U), y(\sigma) = e^{(\sigma-t)A} x + \int_t^\sigma Ae^{(\sigma-s)A} Bu(s) \, ds \right\}$$

we can prove

(15) PROPOSITION. *Let* $P(t)$ *be defined by* (12) *for* $P_0 = 0$; *then*

$$J(t,x) = \langle P(T-t)x, x\rangle_H \quad \forall x \in H, \quad \forall t \in [0, T_1].$$

We note that the "a priori" estimation of proposition (11) holds if and only if $\| P(t)Ax \|_H \le c \| x \|_{D_A(\alpha,\infty)}$ $\forall x \in D(A)$ where c does not depend on t.

3. GLOBAL EXISTENCE

We use (15) to prove the global existence for (8); then we study (8) with

$$Q_0 = 0 . \tag{16}$$

In order to prove an " a priori " estimation for the maximal solution of (8), we consider the following family of optimal control problems:

$$\begin{cases} \text{minimize} \quad J_s(u) = \int_s^T \{ \langle My(\sigma), y(\sigma)\rangle_H + \langle Nu(\sigma), u(\sigma)\rangle_U \} \, d\sigma \\ \text{over all} \quad u \in L^2(s,T;U), \text{ where } y(\sigma) = e^{(\sigma-s)A} x + \int_s^\sigma A e^{(\sigma-\tau)A} Bu(\tau)\, d\tau. \end{cases} \tag{17}$$

Problem (17) admits a unique solution; optimality conditions have been found, for example in [2]. For every $s \in [0, T]$ there exists $\tilde{P}(s) \in \Sigma^+(H)$ such that $J(s, x) = \langle \tilde{P}(s) x, x\rangle_H$; denoting by $y^\star_{(s,x)}$ the optimal trajectory of (17), we have

$$\langle \tilde{P}(s) x_1, x_2\rangle = \int_t^T \langle M y^\star_{(s,x_1)}(\sigma), e^{(\sigma-s)A} x_2\rangle \, d\sigma \qquad \forall x_1, x_2 \in H . \tag{18}$$

We can prove the following lemma on the regularity of $y^\star_{(s,x_1)}$:

(19) LEMMA. $\forall s \in [0, T]$, $\forall x_1 \in H$, $y^\star_{(s,x_1)}$ is a continuous function from $[s, T]$ to H, and there exists $c > 0$ such that

$$\| y^\star_{(s;x_1)}(\sigma) \|_H \leq c \| x_1 \|_H \qquad \forall s \in [0, T], \quad \forall \sigma \in [s, T], \quad \forall x_1 \in H.$$

In the proof of (20) we have used (9), and a theorem by [6]. We evaluate (18) for $x_2 = Ax$, $x \in D(A)$; from (19), (9) and (15) we have

(20) PROPOSITION. *Let* Q *be the maximal solution of* (8) *with* $Q_0 = 0$. *Then there exists* $c > 0$ *such that* $\| Q(t) \|_{\mathscr{L}(D_A(\alpha,\infty);H)} \leq c$ *for every* t *in the interval of existence. Therefore* (8) *has a global solution* $Q \in C_F([0,T]; \mathscr{L}(D_A(\alpha,\infty); H))$.

Finally, using $Q(t)$ given by (20), we may prove that the optimal control of problems (5), (6) is

$$u^\star(t) = - N^{-1} (Q(t) B)^\star y^\star(t)$$

where $y^*(t)$ is the unique solution of

$$y^\star(t) = e^{tA} x - \int_0^t A e^{(t-s)A} BN^{-1} (Q(s) B)^\star y^\star(s) \, ds .$$

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