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Regularity properties of a stochastic convolution integral

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Analisi matematica. — *Regularity properties of a stochastic convolution integral.* Nota di GIUSEPPE DA PRATO, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studiano proprietà di regolarità di un integrale di convoluzione del tipo Itô.

1. INTRODUCTION

Consider the following Itô integral

$$u(t) = \int_0^t e^{(t-s)A} f(s) \, dW_s = I(f) . \tag{1}$$

Here e^{tA} is an analytic semigroup on the Hilbert space H, W_s is a real Brownian motion and f belongs to $M_W^2(0, T; H)$, the Hilbert space of all non anticipative H-valued stochastic processes f such that

$$E \int_0^t \|f(s)\|^2 \, ds < +\infty . \tag{2}$$

In this paper we show that, under suitable assumptions, u belongs to $M_W^2(0, T; D_A(1/2, 2))$ where $D_A(1/2, 2)$ is the Lions-Peetre interpolation space $(D_A, H)_{1/2,2}$ (see [2], [4]). We recall that if A is self-adjoint negative then $D_A(1/2, 2)$ reduces to $D((-A)^{1/2})$.

The above result is useful for studying the following stochastic differential equation

$$\begin{cases} du = Au \, dt + Bu \, dW_t \\ u(0) = u_0 \end{cases} \tag{3}$$

which we write in the following mild form

$$u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} Bu(s) \, dW_s = Z(u) . \tag{4}$$

(*) Nella seduta del 24 aprile 1982.

In fact if B maps $D_A(1/2, 2)$ into H (as for instance when A is a second order and B a first order differential operator) then Z maps $M_W^2(0, T; D_A(1/2, 2))$ into itself and we may use the contractions principle in order to solve (4). We shall give some results in this direction in a next paper.

2. The result

We recall the following characterization of $D_A(\theta, 2)$, (see [1])

$$D_A(\theta, 2) = \{x \in H ; h^{1-\theta} A e^{hA} x \in L_*^2(\mathbf{R}_+) \tag{5}$$

and we set

$$\|x\|_{D_A(\theta,2)}^2 = \int_0^\infty \|h^{1-\theta} A e^{hA} x\|^2 \, dh/h . \tag{6}$$

Here $L_*^2(\mathbf{R}_+)$ represents the set of all real functions in $\mathbf{R}_+$ which are square integrable with respect to the measure dt/t.

Proposition 1. *Assume that* $f \in M_W^2(0, T; D_A(q, 2))$, $q \in]0, 1/2[$, *and let* $u = I(f)$ *be defined by* (1). *Then we have* $u \in M_W^2(0, T; D_A(q + 1/2, 2))$ and $\|I\| \le (1 - q)^{-1}$.

Proof. Recalling the properties of the Ito integral (see for instance [5]) we have for any $\theta \in]0, 1[$

$$E\|Ah^{1-\theta} e^{hA} u(t)\|^2 = E \int_0^t \|h^{1-\theta} A e^{(t-s+h)A} f(s)\|^2 \, ds = \tag{7}$$

$$= E \int_0^\infty \|h^{1-\theta} A e^{(s+h)A} \tilde{f}(t-s)\|^2 \, ds$$

where $\tilde{f}(t) = f(t)$ if $t \in [0, T]$ and $\tilde{f}(t) = 0$ otherwise. It follows

$$K = E \int_0^T dt \int_0^\infty \|Ah^{1-\theta} e^{hA} u(t)\|^2 \, dh/h = \tag{8}$$

$$= E \int_0^\infty dh/h \int_0^\infty ds \int_0^T \|h^{1-\theta} A e^{(s+h)A} \tilde{f}(t-s)\|^2 \, dt \le$$

$$\le E \int_0^\infty ds \int_0^T dt \int_0^\infty (h^{2-2\theta} (s+h)^{2q-2} \|(s+h)^{1-q} A e^{(s+h)A} \tilde{f}(t)\|^2) \, dh/h.$$

Setting $s + h = k$ we get

$$\text{(9)} \qquad K \leq E \int_0^T dt \int_0^\infty dk \int_0^K \| k^{1-q} A e^{kA} \tilde{f}(t) \|^2 (k - s)^{1-2\theta} k^{2q-2} \, ds =$$

$$= (2 - 2\theta)^{-1} E \int_0^T dt \int_0^\infty dk k^{2q-2\theta} \| k^{1-q} A e^{kA} \tilde{f}(t) \|^2$$

which for $\theta = q + 1/2$ yields

$$K = \| u \|^2_{M^2_W(0,T;D_A(1/2+q,2))} \leq (1 - 2q)^{-1} \| f \|^2_{M^2_W(0,T;D_A(q,2))}$$

and the proof is complete.

Remarks 2.

a) Assume that A is self-adjoint and strictly negative, set $B = -A$, $\| x \|_{-q} = \| B^{-q} x \|$ and call $H_q = D(B^{-q})$ the completion of H with the norm $\| \ \|_{-q}$. We have

$$(D(B^{1-q}), D(B^{-q}))_{\theta,2} = D(B^{\theta-q}) = H \qquad \text{if} \quad \theta = q .$$

From Proposition 1 we find that I maps $M^2_W(0, T; H)$ into $M^2_W(0, T; D(B^{1/2}))$ and moreover $\| I \| \leq 1$.

b) If $A = C + D$ with C self-adjoint and D dominated by C (see [3]) we can show, adapting the previous argument, that I maps $M^2_W(0, T; H)$ into $M^2_W(0, T; D(1/2, 2))$, with $\| I \| \leq 1$.

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