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A Riccati equation arising in a boundary control problem for distributed parameters

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Analisi matematica. — *A Riccati equation arising in a boundary control problem for distributed parameters.* Nota di FRANCO FLANDOLI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si prova l'esistenza locale della soluzione di una equazione di Riccati che si incontra in un problema di controllo ottimale. In ipotesi di regolarità per il costo si prova l'esistenza globale. Il problema astratto considerato è il modello di alcuni problemi di controllo ottimale governati da equazioni paraboliche con controllo sulla frontiera.

1. LOCAL EXISTENCE

Let H and U be two Hilbert spaces. We study the Riccati equation

$$(1)\qquad \begin{cases} P'(t) = A^* P(t) + P(t) A + 1 - P(t) ABB^* A^* P(t) & \text{in} \quad [0, T] \\ P(0) = P_0 \end{cases}$$

where $P(t)$, A, P_0 are linear operators on H, A^* is the adjoint of A, and B is a linear bounded operator from U to H. Problem [1] is connected with the synthesis of the following optimal control problem:

$$\text{minimize} \qquad J(u) = \int_0^T \{\| y(t)\|_H^2 + \| u(t)\|_U^2\}\, dt$$

$$\text{over all } u \in L^2(0, T; U)\ ^{(1)}$$

where the state $y(t)$ is defined by the abstract equation in H:

$$(2)\qquad \begin{cases} y'(t) = A(y(t) - Bu(t)) & \text{in} \quad [0, T] \\ y(0) = y_0 \quad , \quad y_0 \in H . \end{cases}$$

We assume that A is the infinitesimal generator of an analytic semigroup, denoted by e^{tA}.

We consider the case in which it is not possible to define the composition AB (as it happens for example in several boundary control problems) and therefore equations (1) and (2) are not well defined.

(*) Nella seduta del 13 marzo 1982.

(1) A function u from $[0, T]$ to U belongs to $L^2(0, T; U)$ if u is Bochner measurable on $[0, T]$ and $\int_0^T \| u(t)\|^2 dt < +\infty$.

Denoting by A^α the fractional powers of the operator A (see for example [1]), suppose that

(3) $\quad \exists \alpha \in (0,1)$ such that the composition $A^\alpha B$ is well defined;

A^α is closed, hence by the closed graph theorem $A^\alpha B$ is a continuous operator from U to H. Since e^{tA} is analytic,

(4) $\quad \exists c > 0$ such that $\|A^{1-\alpha} e^{tA}\| \leq \dfrac{c}{t^{1-\alpha}}$ for $t > 0$, $\alpha \in (0,1)$;

therefore, for every $u \in L^2(0, T; U)$, the function

$$y(t) = e^{tA} y_0 - \int_0^t A e^{(t-s)A} B u(s) \, ds \tag{5}$$

is defined a.e. on $[0, T]$ and $y(\cdot) \in L^2(0, T; H)$; representation (5) of the function y gives a more precise interpretation of equation (2).

We return to equation (1); the operator $Q(t) = P(t) A^{1-\alpha}$ satisfies the "non symmetric" Riccati equation

$$\begin{cases} Q'(t) = A^* Q(t) + Q(t) A + (1 - Q(t) K Q^*(t)) A^{1-\alpha} \\ Q(0) = P_0 A^{1-\alpha} = Q_0 \end{cases} \tag{6}$$

where $K = A^\alpha B (A^\alpha B)^* \in L(H)$ (2). We assume that $Q_0 \in L(H)$. Using the variation of constant device we may write (6) in the integral form

$$Q(t) x = e^{tA^*} Q_0 e^{tA} x + \int_0^t e^{(t-s)A^*} (1 - Q(s) K Q^*(s)) A^{1-\alpha} e^{(t-s)A} x \, ds, \quad x \in H. \tag{7}$$

We study (7) in the space $C_F([0, T]; L(H))$ of the functions $t \to U(t)$, from $[0, T]$ to $L(H)$, such that $U(\cdot) x$ is continuous for every $x \in H$.

(8) Proposition. For a suitable $T_1 > 0$ there exists a unique solution $Q(t)$ of (7) in the space $C_F([0, T_1]; L(H))$. We give a sketch of the proof: let $B([0, T_1]; L(H))$ be the space $C_F([0, T_1]; L(H))$ endowed with the norm $\|U\|_B = \sup_{[0,T]} \|U(t)\|_{L(H)}$; we set $K_\eta = \{U \in B([0, T_1]; L(H)), \|U\|_B \leq \eta\}$, and we define $\gamma : K_\eta \to K_\eta$ as

$$\gamma(U)(t) = e^{tA^*} Q_0 e^{tA} + \int_0^t e^{(t-s)A^*} (1 - U(s) K U^*(s)) A^{1-\alpha} e^{(t-s)A} \, ds;$$

then γ is a contraction and (8) follows from the contraction principle.

(2) L (H) is the space of linear bounded operators from H to H.

The proposition enables us to solve the synthesis problem locally; if T_1 is given by (8), the equation

$$y(t) = e^{(t-T+T_1)A}\, y(T-T_1) - \int_{T-T_1}^{t} Ae^{(t-s)A}\, B\, (A^{\alpha} B)^{*}\, Q^{*}(T-s)\, y(s)\, ds$$

has a solution $y(\cdot) \in L^2(T-T_1, T; H)$, and it can be proved that the feedback control $u(t) = (A^{\alpha} B)^{*}\, Q^{*}(T-t)\, y(t)$ defined on $[T-T_1, T]$ coincides with the optimal control on $[T-T_1, T]$.

2. Global existence

We prove a global existence result for the optimal control problem of minimizing

$$J(u) = \int_0^T \{\|My(t)\|_H^2 + \|u(t)\|_U^2\}\, dt$$

over all $u \in L^2(0, T; U)$, where the state is defined by (2), and M satisfies:

(9) the linear operator $A^{*1-\alpha} M^{*}$ is continuous from H to H;

under this assumption the Riccati equation

$$\begin{cases} S'(t) = A^{*} S(t) + S(t) A + A^{*1-\alpha} M^{*} M A^{1-\alpha} - S(t) K S(t) \\ S(0) = S_0 \end{cases}$$

has a global solution $S(\cdot) \in C_F([0, T]; L(H))$ (see [2]) and $Q(t) = A^{*\alpha-1} S(t)$ is the global solution of

$$Q(t) = e^{tA^{*}} A^{*\alpha-1} S_0\, e^{tA} + \int_0^t e^{(t-s)A^{*}} (M^{*} M - Q(s)\, K Q^{*}(s))\, A^{1-\alpha}\, e^{(t-s)A}\, ds.$$

This global result generalizes a proposition by [3].

3. An example

Let Ω be a bounded open subset of R^n with boundary Γ which is a C^{∞} manifold of dimension $(n-1)$, and let E be the elliptic operator

$$Ef(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\, a_{ij}(x)\, \frac{\partial}{\partial x_j} f(x),$$

where

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \geq \nu\,|\xi|^2, \qquad \nu > 0 \quad , \quad \xi \in \mathrm{R}^n, \qquad \text{a.e. in } \Omega\,,$$

$$a_{ij} \in \mathrm{C}^{\infty}(\overline{\Omega})\,.$$

We consider the parabolic equation

$$(10) \qquad \begin{cases} \dfrac{\partial y}{\partial t} = \mathrm{E}y & \text{in } (0\,,\mathrm{T}) \times \Omega \\ y = u & \text{in } (0\,,\mathrm{T}) \times \Gamma \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

where u is the control. It is known (see [4]) that the linear operator A defined by

$$\begin{cases} \mathrm{D}(\mathrm{A}) = \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega) \\ \mathrm{A}f = \mathrm{E}f \qquad \text{for } f \in \mathrm{D}(\mathrm{A}) \end{cases}$$

generates an analytic semigroup in $\mathrm{L}^2(\Omega)$, and that for $0 \leq \alpha \leq 1/4$ $\mathrm{D}(\mathrm{A}^{\alpha}) = \mathrm{H}^{2\alpha}(\Omega)$; following an idea of [5], we consider the stationary problem

$$(11) \qquad \begin{cases} \mathrm{E}\varphi = 0 & \text{in } \Omega \\ \varphi = g & \text{in } \Gamma\,. \end{cases}$$

In [4] it is proved that the Green's mapping B defined by $\mathrm{B}g = \varphi$ is linear bounded from $\mathrm{L}^2(\Gamma)$ to $\mathrm{H}^{1/2}(\Omega)$, so that $\mathrm{A}^{\alpha}\,\mathrm{B}$ is bounded from $\mathrm{L}^2(\Gamma)$ to $\mathrm{L}^2(\Omega)$ for $0 \leq \alpha \leq 1/4$.

It may be verified that the mild solution (5) is a natural generalized solution of problem (10) for all controls $u \in \mathrm{L}^2(0\,,\mathrm{T}\,;\mathrm{L}^2(\Gamma))$.

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