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Semicontinuity in $L^\infty$ for polyconvex integrals


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Calcolo delle variazioni. — *Semicontinuity in* $L^\infty$ *for polyconvex integrals* (*). Nota di EMILIO ACERBI (**), GIUSEPPE BUTTAZZO (**), and NICOLA FUSCO (***) presented (*) by Socio C. MIRANDA.

**Riassunto.** — Viene studiata la semicontinuità rispetto alla topologia di $L^\infty$ ($\Omega ; \mathbb{R}^m$) per alcuni funzionali del Calcolo delle Variazioni dipendenti da funzioni a valori vettoriali.

**INTRODUCTION**

The first results about the semicontinuity of functionals of the type

\begin{equation}
\int_\Omega f(x, u(x), Du(x)) \, dx,
\end{equation}

where $u$ is a vector-valued function, are due to Morrey [9] who proved that under certain regularity assumptions on the integrand $f$, the functional (1) is weakly* sequentially l.s. (lower semicontinuous) on $W^{1,\infty} (\Omega ; \mathbb{R}^m)$ if and only if for all $(x, s)$ the function $\xi \to f(x, s, \xi)$ is quasi-convex:

**DEFINITION 1.** A continuous function $\phi : \mathbb{R}^{nm} \to \mathbb{R}$ is quasi-convex if for every open subset $\Omega$ of $\mathbb{R}^n$, for every $\xi \in \mathbb{R}^{nm}$ and every function $w \in C^1_0 (\Omega ; \mathbb{R}^m)$

\[ \phi (\xi) \operatorname{meas} (\Omega) \leq \int_\Omega \phi (\xi + Dw (x)) \, dx. \]

The result of Morrey was generalized by Meyers [8] (in the case of integrals of any order) and by Acerbi and Fusco [1] who showed that if $f$ is a Carathéodory function such that

\begin{equation}
0 \leq f(x, s, \xi) \leq a(x) + C (|s|^p + |\xi|^p)
\end{equation}

($p \geq 1$),

where $a$ is non-negative and locally summable on $\mathbb{R}^n$, and $C > 0$, then the functional (1) is weakly l.s. on $W^{1,p} (\Omega ; \mathbb{R}^m)$ if and only if the function $\xi \to f(x, s, \xi)$ is quasi-convex.

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Finally, we remark that several semicontinuity theorems have been also proved for convex functionals, even more general than the functional (1) (see e.g. [6]).

In the applications to nonlinear elasticity, one usually finds a particular class of quasi-convex functions, namely the class of polyconvex functions (see Ball [2], [3], Dacorogna [5]):

**Definition 2.** A function \( \phi : \mathbb{R}^{mn} \to \mathbb{R} \) is polyconvex if there exists a convex function \( \psi \) such that for every \( m \times n \) matrix \( A \)

\[
\phi(A) = \psi(XA),
\]

where \( XA \) denotes the vector whose components consist of all the subdeterminants of the matrix \( A \).

We remark here that if \( \xi \to f(x, s, \xi) \) is for all \((x, s)\) a non-negative polyconvex function, the semicontinuity theorem proved in [1] holds without the growth condition (2).

In general, if \( 1 \leq p < \infty \), the integrals of polyconvex functions are not l.s. with respect to the topology of \( L^p(\Omega, \mathbb{R}^m) \) as one can already see for the functional \( (m = n) \)

\[
\int_{\Omega} |\det D\phi(x)| \, dx.
\]

For it is possible to construct a sequence \((u_h)\) such that \( \det D\phi = 0 \), but \((u_h)\) converges in \( L^p \), for every \( p < \infty \), to the function \( u(x) = x \). Nevertheless this functional turns out to be l.s. with respect to the topology of \( L^\infty_{\text{loc}}(\Omega; \mathbb{R}^n) \).

This example shows why for the integrals of polyconvex functions it is not interesting to study the lower semicontinuity with respect to the topology of \( L^p(\Omega, \mathbb{R}^m) \) (with \( p < \infty \)).

On this subject the situation in the scalar case is completely different. Indeed when \( u \) is a real-valued function and \( f \) satisfies (2), the semicontinuity of the functional (1) with respect to the topology of \( L^\infty(\Omega) \) is equivalent to the semicontinuity in the topology of \( L^p(\Omega) \), at least on \( W^{1,p}(\Omega) \cap L^\infty(\Omega) \) (see Carbone-Sbordone [4]).

In this paper we state some results on the semicontinuity in the topology of \( L^\infty_{\text{loc}}(\Omega; \mathbb{R}^n) \) of certain integrals of polyconvex functions.

By the previous remark it is clear that the results stated here generalize the semicontinuity theorems proved in the scalar case in [10], [11]. At any rate they apply to the important class of parametric integrals (widely studied in [7], [10]) and to the most representative examples of polyconvex integrals occurring in nonlinear elasticity.
Let \( \Omega \) be an open subset of \( \mathbb{R}^n \); if \( u \) is a function from \( \Omega \) to \( \mathbb{R}^m \), let \( Du \) denote the matrix of its derivatives, and let \( X^0 u \) denote the vector whose \( \binom{m}{n} \) components are the subdeterminants of \( Du \) of order \( n \). In the following theorem we specialize to the case \( m = n + 1 \).

**Theorem 1.** Let \( f: \Omega \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to [0, + \infty) \) satisfy:

1. for every \((x, s) \in \Omega \times \mathbb{R}^{n+1}\) the function \( \xi \mapsto f(x, s, \xi) \) is convex and lower semicontinuous, and \( f(x, s, 0) = 0 \);
2. for every \( \Sigma \subset \subset \Omega \times \mathbb{R}^{n+1} \) there exists a continuous function \( \omega_\Sigma: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \), vanishing in \((0, 0)\), such that for every \((x, s), (y, t) \in \Sigma \) and every \( \xi \in \mathbb{R}^{n+1} \)

\[
| f(x, s, \xi) - f(y, t, \xi) | \leq \omega_\Sigma (|x - y|, |s - t|) [1 + f(x, s, \xi)].
\]

Then the functional \( \int f(x, u(x), X^0 u(x)) \, dx \) is l.s. on the space \( W^{1,n}_\text{loc} (\Omega; \mathbb{R}^{n+1}) \cap C(\Omega; \mathbb{R}^{n+1}) \), endowed with the topology of \( L^\infty_\text{loc} (\Omega; \mathbb{R}^{n+1}) \).

An interesting example of integrand satisfying (1), (2) is

\[
f(x, s, \xi) = a(x, s) \phi(\xi)
\]

where \( \phi \) is convex and lower semicontinuous, \( \phi(0) = 0 \), and \( a \) is a continuous function with some positive lower bound.

We remark that the foregoing result is still valid for the functional

\[
\int f(x, u(x), \det Du(x)) \, dx,
\]

where \( u \) is a function from \( \Omega \) to \( \mathbb{R}^n \) and \( f: \Omega \times \mathbb{R}^n \times \mathbb{R} \to [0, + \infty] \) satisfies conditions analogous to (1), (2).

More generally, one may consider functionals of the type

\[
\int f(x, u(x), X^+ u(x)) \, dx,
\]

where \( X^+ u \) denotes the vector whose components are the absolute values of all the subdeterminants of \( Du \); in addition let \( r(n, m) \) denote the dimension of the vector \( X^+ u \).

**Theorem 2.** Let \( f: \Omega \times \mathbb{R}^m \times \mathbb{R}^{r(n,m)}_+ \to [0, + \infty] \) satisfy:

1. for every \((x, s) \in \Omega \times \mathbb{R}^m \) the function \( \xi \mapsto f(x, s, \xi_+) \) is convex and lower semicontinuous (here \( \xi_+ \) denotes the vector of the absolute values of the components of \( \xi \)).
for every $\Sigma \subset \subset \Omega \times R^m$ there exists a continuous function $\omega_\Sigma : R_+ \times R_+ \to R_+$, vanishing in $(0, 0)$, such that for every $(x, s), (y, t) \in \Sigma$ and for every $\xi_+ \in R^{(n,m)}_+$

$$|f(x, s, \xi_+) - f(y, t, \xi_+)| \leq \omega_\Sigma(|x - y|, |s - t|) [1 + f(x, s, \xi_+)].$$  

Then the functional $\int f(x, u(x), X^+ u(x)) \, dx$ is l.s. on the space $W^{1,n}_{loc} (\Omega ; R^m) \cap C(\Omega ; R^m)$, endowed with the topology of $L^\infty_{loc}(\Omega ; R^m)$.

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