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Abstract singular hyperbolic equations

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Analisi matematica. — *Abstract singular hyperbolic equations.* Nota di GIUSEPPE COPPOLETTA, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si annunziano alcuni risultati di esistenza e unicità per l'equazione astratta singolare

$$\varphi(t)\, u'(t) = A(t)\, u(t) + f(t)$$

nel caso iperbolico.

INTRODUCTION

Let E be a Banach space (norm $|\cdot|_E$) and let $\{A(t)\}_{t\in[0,T]}$ be a family of linear operators $A(t) : D(t) \subseteq E \to E$ with dense domain in E.

Consider the problem

$$\text{(P)} \qquad \varphi(t)\, u'(t) = A(t)\, u(t) + f(t) \qquad t \in]0, T]$$

where $\varphi :]0, T] \to [0, +\infty]$ is an arbitrary function such that:

$$0 \le \varphi(t) < +\infty \qquad \text{a.e. in }]0, T]$$

$$\frac{1}{\varphi} \in L^1_{loc}(]0, T]).$$

This formulation includes many kinds of singular and degenerate evolution equations which have been extensively studied, essentially in the parabolic framework (see for example [1], [6], [8], [10]).

In this note we study problem (P) in the hyperbolic case (see also [2], [3]). In section 1 we consider the case $\varphi(t) = 1$ and, by relaxing the assumptions on $\{A(t)\}$, we generalize the results of Da Prato–Grisvard [5], and Da Prato–Iannelli [7] for the classical evolution equation

$$(P_1) \qquad \begin{cases} u'(t) = A(t)\, u(t) + f(t) & t \in [0, T]. \\ u(0) = x \end{cases}$$

In section 2 we study the case $\varphi(t) = t$ and obtain existence and uniqueness results in an appropriate weighted space for the singular abstract equation

$$(P_2) \qquad t u'(t) = A(t)\, u(t) + f(t) \qquad t \in]0, T].$$

A priori no initial condition can be imposed here.

Finally, in section 3, the general problem (P) is reduced to cases (P_1) and (P_2) (depending on $\varphi(t)$) by means of suitable change of variables.

This note will appear in a more detailed form in a forthcoming paper.

(*) Nella seduta del 9 gennaio 1982.

0. NOTATIONS

If X is a Banach space and $L : D(L) \subseteq X \to X$ is a linear operator, we denote by $\rho(L)$ the resolvent set of L and by $R(\lambda, L)$ the corresponding resolvent operator. Let $\{A(t)\}_{t \in [0,T]}$ be a family of linear operators $A(t) : D(t) \subseteq E \to E$. The family is said to be ω-measurable (1) if there exists $\omega \in R$ such that $]\omega, +\infty[\subseteq \rho(A(t))$ for any $t \in [0, T]$ and the mapping $t \to R(\lambda, A(t))\, x$ is measurable for any $x \in E$ and for any $\lambda \in]\omega, +\infty[$. On the other hand the family is said to be (M, ω)-stable (2) if there exist $\omega \in R$ and $M > 0$ such that:

$$| R(\lambda, A(t_1))\, R(\lambda, A(t_2)), \cdots, R(\lambda, A(t_k)) |_{\mathscr{L}(E)} \leq M(\lambda - \omega)^{-k}$$

for any $\lambda > \omega$, $k \in N$ and $0 \leq t_1 \leq t_2 \leq, \cdots, \leq t_k \leq T$.

In this note we always assume the following conditions:

(H)

i) E is reflexive and $\{A(t)\}$ is ω-measurable and (M, ω)-stable in E, $(\omega \in R, M > 0)$

ii) there exists a Banach space $F \underset{ds}{\hookrightarrow} E$ (3) such that $F \subseteq D(t)$ for any $t \in [0, T]$ and $\{A(t)_F\}$ (4) is η-measurable and (N, η) stable in F, $(\eta \in R, N > 0)$.

iii) the map $t \to |A(t)|_{\mathscr{L}(F,E)}$ is measurable.

Now let $X \subseteq L^1_{loc}(]0, T[, E)$ be a Banach space; consider the following linear operators on X:

$$(1) \quad \begin{cases} D(A) = \{u \in X \mid u(t) \in D(t) \quad \text{a.e. in }]0, T[\text{ and } t \to A(t)\, u(t) \in X\} \\ (Au)(t) = A(t)\, u(t) \end{cases}$$

$$(2) \quad \begin{cases} D(B) = \{u \in X \mid \varphi(t)\, u'(t) \in X\} \ ^{(5)} \\ (Bu)(t) = -\varphi(t)\, u'(t). \end{cases}$$

Then we can write (P) in the following form:

$$(3) \quad Bu + Au = -f.$$

(1) Cfr. [5].
(2) Cfr. [9].
(3) The simbol $\underset{ds}{\hookrightarrow}$ denote continuous and dense imbedding.
(4) $A(t)_F$ denotes the part of $A(t)$ in F, i.e. the operator
$D(A(t)_F) = \{x \in D(t) \cap F \mid A(t)\, x \in F\}$, $(A(t)_F)\, x = A(t)\, x \quad \forall x \in D(A(t)_F)$.
(5) The derivative is taken in the distributional sense.

We say that u is a strict solution of (3) in X if $u \in D(B) \cap D(A)$ and u satisfies (3). On the other hand u is said to be a strong solution of (3) in X if there exists a sequence $\{v_k\} \subseteq D(B) \cap D(A)$ such that

$$v_k \xrightarrow{X} u \qquad \text{and} \quad Bv_k + Av_k \xrightarrow{X} -f .$$

We shall see that, depending on $\varphi(t)$, an initial condition may or may not be needed for uniqueness. In the former case we also require in the above definitions the condition $u(0) = x$ for strict solutions and $v_k(0) \xrightarrow{E} x$ for strong solutions.

1. The case $\varphi(t) = 1$

In this section we solve the abstract Cauchy problem:

$$(P_1) \qquad \begin{cases} u'(t) = A(t)\,u(t) + f(t) & t \in [0, T] \\ u(0) = x \end{cases}$$

To reach an adeguate generality in the study of time-singularities for the general problem (P) we find it convenient to relax the following condition which has been assumed in [5] and [7]:

β) hypothesis (H) is verified and there exists $\beta > 0$ such that

$$|A(t)|_{\mathscr{L}(F,E)} \leq \beta \qquad \forall t \in [0, T].$$

More precisely we obtain the following:

THEOREM 1. *Assume that* (H) *is verified and that*

$$(4) \qquad \int_0^T |A(t)|^p_{\mathscr{L}(F,E)}\, dt < +\infty .$$

Then, for any $p \in [1, +\infty[$, $x \in E$ *and* $f \in L^p(0, T, E)$, *problem* (P_1) *has a unique strong solution* u *in* $L^p(0, T, E)$ *such that* $u \in C([0, T], E)$ *and* $u(0) = x$.

Moreover, if u_n *is the solution of the Yosida approximating problem* [(6)]

$$(P'_n) \qquad \begin{cases} u'_n(t) = A_n(t)\,u_n(t) + f(t) & t \in [0, T] \\ u_n(0) = x \end{cases}$$

then $u_n \to u$ *in* $C([0, T], E)$.

Finally, if $x \in F$ *and* $f \in L^p(0, T, F)$, *then the solution* u *is strict; if further* F *is reflexive, then* $u \in L^\infty(0, T, F)$.

(6) $A_n(t) = n^2 R(n, A(t)) - nI$.

Remark 1. If we only look for solutions of (P_1) which vanish for $t = 0$, then we can replace (4) by the following condition:

$$\int_{\varepsilon}^{T} |A(t)|^{p}_{\mathcal{L}(F,E)}\, dt < +\infty \qquad \forall \varepsilon \in]0, T[. \tag{5}$$

On the contrary, if $u(0) = x \neq 0$, we cannot simply drop the condition (4).

2. THE CASE $\varphi(t) = t$

Here we study the equation

$$tu'(t) = A(t)\, u(t) + f(t) \qquad t \in]0, T]. \tag{P_2}$$

For any $\alpha \in R$ set:

$$X_\alpha = \{u :]0, T[\to E \mid t^{-\alpha} u(t) \in L^p(0, T, E)\}.$$

THEOREM 2. *Assume that* (H) *and* (5) *are verified.*
Then, for any $p \in [1, +\infty[$, $\alpha > \omega + \dfrac{1}{p}$ *and for any* $f \in X_\alpha$ *equation* (P_2) *has a unique strong solution* u *in* X_α *such that*

$$t^{-\alpha+(1/p)} u(t) \in C([0, T], E) \quad \textit{and} \quad \lim_{t\to 0} t^{-\alpha+(1/p)} u(t) = 0. \tag{6}$$

Moreover, if u_n *is the solution of the Yosida approximating problem*

$$tu_n'(t) = A_n(t)\, u_n(t) + f(t) \qquad t \in]0, T], \tag{P_2^n}$$

then

$$t^{-\alpha+(1/p)} u_n(t) \xrightarrow[n\to\infty]{} t^{-\alpha+(1/p)} u(t) \quad \text{in} \quad C([0, T], E).$$

Finally, if (4) *is verified, if* F *is reflexive,* $\alpha > \eta + \dfrac{1}{p}$, $p > 1$, *and if* $t^{-\alpha} f(t) \in L^\infty(0, T, F)$, *then the solution* u *is strict and*

$$t^{-\alpha} u(t) \in L^\infty(0, T, F).$$

Remark 2. Observe that (6) gives an implicit " initial trend " of the solution u; this make clear why one cannot a priori require an arbitrary initial condition.

3. The general case

Let $\varphi :]0\,,T] \to [0\,,+\infty]$ be an arbitrary function such that

$$(7) \qquad \begin{cases} 0 \le \varphi(t) < +\infty & \text{a.e. in }]0\,,T[\\ \dfrac{1}{\varphi} \in L^1_{loc}(]0\,,T]). \end{cases}$$

We distinguish two subcases:

$$\int_0^T \frac{ds}{\varphi(s)} < +\infty \qquad \text{and} \qquad \int_0^T \frac{ds}{\varphi(s)} = +\infty\,.$$

In the first subcase we solve the abstract Cauchy problem:

$$(P_c) \qquad \begin{cases} \varphi(t)\,u'(t) = A(t)\,u(t) + f(t) & t \in]0\,,T] \\ u(0) = x\,. \end{cases}$$

Theorem 3. *Assume that:*

i) $\displaystyle\int_0^T \frac{ds}{\varphi(s)} < +\infty$ *and* (7) *is verified*

ii) the family $\{A(t)\}$ *verifies* (H) *and*

$$\int_0^T \frac{|A(t)|^p_{\mathcal{L}(F,E)}}{\varphi(t)}\,dt < +\infty\,.$$

Let μ *denote the positive measure defined by:*

$$\frac{d\mu}{dt} = \frac{1}{\varphi(t)}\,.$$

Then, for any $p \in [1\,,+\infty[\,,\ x \in E$ *and* $f \in L^p(0\,,T\,,\mu\,,E)$ (7) *problem* (P_c) *has a unique strong solution* u *in* $L^p(0\,,T\,,\mu\,,E)$ *such that* $u \in C([0\,,T]\,,E)$ *and* $u(0) = x$.

Further results, analogous to those obtained in Theorem 1, are also valid if $x \in F\,, f \in L^p(0\,,T\,,\mu\,,F)$ *and if* F *is reflexive.*

The proof is obtained by reducing (P_c) to the case (P_1) with the transformation

$$u(t) = v\left(\int_0^t \frac{ds}{\varphi(s)}\right).$$

(7) This is the space of all functions $f :]0\,,T[\to E$ such that $|f(\cdot)|^p_E$ is μ-integrable.

In the second subcase we solve (P) without any initial condition:

THEOREM 4. *Assume that*:

i) $\int_0^T \frac{ds}{\varphi(s)} = +\infty$ *and* (7) *is verified*

ii) *the family* $\{A(t)\}$ *satisfies* (H) *and*

$$\int_\varepsilon^T \frac{|A(t)|^p_{\mathcal{L}(F,E)}}{\varphi(t)}\, dt < +\infty, \qquad \forall \varepsilon \in]0, T[$$

Let μ *denote the positive measure defined by*

$$\frac{d\mu}{dt} = \frac{1}{\varphi(t)} \exp\left[(\alpha p - 1)\int_t^T \frac{ds}{\varphi(s)}\right],$$

with $p \in [1, +\infty[$ *and* $\alpha > \omega + \frac{1}{p}$.

Then, for any $f \in L^p(0, T, \mu, E)$, *problem* (P) *has a unique strong solution* u *in* $L^p(0, T, \mu, E)$ *such that*

$$u(t) \exp\left[\left(\alpha - \frac{1}{p}\right)\int_t^T \frac{ds}{\varphi(s)}\right] \in C([0, T], E),$$

$$\lim_{t\to 0} u(t) \exp\left[\left(\alpha - \frac{1}{p}\right)\int_t^T \frac{ds}{\varphi(s)}\right] = 0.$$

Once again further results, analogous to the corresponding ones of Theorem 2, are also valid.

The proof makes use of the following transformation

$$u(t) = v\left(\exp\left(-\int_t^T \frac{ds}{\varphi(s)}\right)\right)$$

which reduces (P) to the case (P_2).

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