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Pasquale Renno

On an inversion formula of certain Laplace transforms in dissipative wave propagation

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Fisica-matematica. — *On an inversion formula of certain Laplace transforms in dissipative wave propagation* (*). Nota di PASQUALE RENNO, presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — Si determina una formula di inversione di alcune trasformate di Laplace che intervengono nell'analisi formale di problemi al contorno relativi ad una classe di mezzi dissipativi. Le espressioni esplicite proposte definiscono funzioni analitiche a decrescenza rapida dotate di numerose proprietà di massimo, utili anche all'analisi di problemi unilaterali.

0. – The inversion of Laplace transforms that occur in the *formal* analysis of problems related to the linear wave propagation in dissipative media is often a problematic question. A typical, noticeable example is given by the dynamic equations of isothermal, isotropic, linear viscoelasticity for threedimensional motions [1]

$$\mathcal{M}\boldsymbol{u} \equiv \int_{-\infty}^{t} [\lambda(t-\tau) + \mu(t-\tau)]\, \partial_\tau \nabla\nabla\cdot \boldsymbol{u}\, d\tau + \int_{-\infty}^{t} \mu(t-\tau)\, \partial_\tau \nabla^2 \boldsymbol{u}\, d\tau - \rho\, \partial_t^2 \boldsymbol{u} = -\boldsymbol{f}(x, t) \tag{0.1}$$

where $\boldsymbol{u}(x, t)$ $(x \in \mathrm{R}^3, t \geq 0)$ is the displacement field, ρ denotes the mass density, while $\lambda(t)$ and $\mu(t)$ are the appropriate relaxation functions. The vector $\boldsymbol{f}$ represents a volume density of prescribed forces.

Even though memory functions as those typical of a standard linear solid are considered (n. 1), one obtains a class of Laplace transforms which is anything but a simple task. In this case, in fact, the fundamental solution E_3 (or E_1) of the threedimensional (or unidimensional) $\mathcal{M}$ operator, or the kernel E_0 which resolves the half-space problem related to (0.1), are *formally* defined by symbolic relations such as (see [2] p. 19):

$$\mathcal{L}_t E_j(x, t) = \hat{g}_j(x, s)\, e^{-k|x|s\sqrt{\frac{s+\beta}{s+\alpha}}}, \qquad (j = 0, 1, 3) \tag{0.2}$$

where $\mathcal{L}_t$ is the Laplace operator with respect to time and s is the parameter of the $\mathcal{L}$-transformation. Furthermore $\hat{g}_j$ is an algebraic $\mathcal{L}$-transform function

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which depends on the boundary-value problem in question and α, β, $k\,(>0)$ are appropriate constants.

Formulae such as (0.2) occur in linearized thermochemistry also [3].

Numerous *formal* and approximate evaluations of E_j by means of the steepest-descent methods are already known (see e.g. [3]), but they are not rigorous and do not give an exhaustive behaviour of the original function. On the other hand, the computation of E_j by means of series expansions or integral representations often leads to very untractable expressions [4].

Recently, [5, 6], we succeeded in obtaining an explicit rigorous inversion formula for the fundamental solution E_3 in terms of a definite integral of modified Bessel functions which has numerous basic properties; so, e.g. we recall that E_3 is a C^∞ rapidly decreasing and never negative function and that the associated distribution is a tempered positive Radon measure [5]. By means of these properties, the distributional and classical initial value problems in the threedimensional case were rigorously discussed in [5].

The aim of this note is to generalize the inversion formula stated in [5], in order to obtain other numerous useful inverse transforms of (0.2) according to various $\hat{g}_j$'s. So, the half-space problem, the unidimensional Cauchy problem, as well as other boundary value problems for a standard linear solid, can be explicitly solved. Furthermore, some remarkable ***maximum properties*** of the fundamental solution E_1 related to the unidimensional case are enunciated (n. 2). On this subject we observe that the explicit expression for E_1 characterizes the *Riemann function* which solves the *general* Cauchy problem when the initial data are prefixed on an arbitrary curve $t=t(x)$ of the xt-plane. Consequently, by means of these properties of E_1 one obtains the explicit solution of the general Cauchy problem and appropriate maximum principles which are of primary importance in order to analyse also unilateral problems for the equation we deal with [7].

1. – If the Helmholtz resolution is employed to represent the displacement vector as

$$\boldsymbol{u} = \nabla \times \nabla + \nabla \psi , \tag{1.1}$$

where $\psi(x, t)$ ($x \in R^3$) is a scalar potential and $\mathbf{A}(x, t)$ is a vector potential, then to solve (0.1) it suffices to analyse an equation such as

$$\int_{-\infty}^{t} g(t-\tau)\, \partial_\tau \nabla^2 \boldsymbol{v}\, d\tau + \boldsymbol{f}(x, t) = \rho \partial_t^2 \boldsymbol{v} . \tag{1.2}$$

If $\boldsymbol{v} = \nabla \times \mathbf{A}$ then $g = \lambda + 2\mu$; when $\boldsymbol{v} = \nabla \psi$ then $g = \mu$. Obviously, attention must be paid to the boundary-initial data and to the source term $\boldsymbol{f}$.

Thus, as it is well known [1], we see that the scalar potential ψ governs the propagation of irrotational waves while the vector potential $\mathbf{A}$ governs the propagation of shear waves.

Consider now a standard linear solid whose relaxation function is

$$g(t) = g(\infty) + [g(0) - g(\infty)]\, e^{-t/\varepsilon}, \tag{1.3}$$

where ε is a relaxation time and $g(\infty) < g(0)$. By (1.2)–(1.3) one has

$$\varepsilon \partial_t (\partial_t^2 \boldsymbol{v} - a_1^2 \nabla^2 \boldsymbol{v}) + \partial_t^2 \boldsymbol{v} - a_0^2 \nabla^2 \boldsymbol{v} = \boldsymbol{f}_* (x, t) \tag{1.4}$$

with $\boldsymbol{f}_* = \rho^{-1} (\varepsilon \partial_t + 1) \boldsymbol{f}$ and

$$a_1^2 = \rho^{-1} g(0) \quad , \quad a_0^2 = \rho^{-1} g(\infty), \qquad (a_0^2 < a_1^2). \tag{1.5}$$

The equation (1.4) governs also the propagation of acoustic waves in a chemically reacting mixture of real gases [3]; in that case a_1 is the frozen sound speed, a_0 is the equilibrium sound speed, while ε is a characteristic time of the chemical reaction. However, in all the usual physical systems $a_0^2 < a_1^2$ results.

Without loss of generality, the strictly hyperbolic operator of (1.4) can be always given the form

$$L_3 = \varepsilon \partial_t (\partial_t^2 - \nabla^2) + \partial_t^2 - c^2 \nabla^2, \tag{1.6}$$

with $c^2 = a_0^2/a_1^2 < 1$. In the unidimensional case, obviously it is

$$L_1 = \varepsilon \partial_t (\partial_t^2 - \partial_x^2) + \partial_t^2 - c^2 \partial_x^2. \tag{1.7}$$

By means of the Fourier and Laplace's operators, it is easy to verify that the fundamental solutions E_3 and E_1 of L_3 and L_1 are *formally* defined by the symbolic relations ($|x| = r$):

$$\hat{E}_3(x, s) = \frac{e^{-r\sigma}}{4\pi r(\varepsilon s + c^2)} \quad , \quad \hat{E}_1(x, s) = \frac{e^{-r\sigma}}{2\sigma(\varepsilon s + c^2)} \tag{1.8}$$

with $\sigma = s\,[(\varepsilon s + 1)/(\varepsilon s + c^2)]^{\frac{1}{2}}$, and $\hat{E}_j = \mathscr{L}_t E_j$. Furthermore, when the half-space (or radiation) problem is considered, the inversion formula for

$$\hat{F}_0(r, s) = e^{-r\sigma} - e^{-r[s + (1-c^2)/2\varepsilon]} \tag{1.9}$$

must be established.

Therefore, to solve explicitly these problems as well as other boundary value problems related to the L_3 and L_1 operators, we now generalize an inversion formula stated in [5].

2. - Let α, β, μ, ν be four arbitrary (real or complex) parameters and $r \in [0, t]$ a real variable; let I_μ denote the modified Bessel function of the first kind. For brevity we put:

$$\frac{\beta + \alpha}{2\varepsilon} = \gamma \quad , \quad \frac{\beta - \alpha}{2\varepsilon} = b \quad , \quad \frac{\alpha}{\varepsilon} = \delta \quad , \quad b\delta = a \tag{2.1}$$

$$F_{\mu,\nu}(r, t) = e^{-\gamma t + \delta r} \int_0^{t-r} e^{(b/2)u} f_{\mu,\nu}(r, t, u)\, du, \tag{2.2}$$

with

$$(2.3)\quad f_{\mu,\nu}=\left(\sqrt{\frac{u}{ar}}\right)^{\mu} \mathrm{I}_{\mu}(2\sqrt{aru})\left(\sqrt{\frac{t-r-u}{t-r}}\right)^{\nu} \mathrm{I}_{\nu}(b\sqrt{(t+r)(t-r-u)}).$$

If

$$(2.4)\qquad \Omega \equiv \{(r,t)\in \mathrm{R}^2 : 0<r<t\ ,\ t>0\},$$

the following lemma holds:

LEMMA 2.1. $\forall(\mu,\nu)$, *the* $\mathrm{F}_{\mu,\nu}$*'s are* $\mathrm{C}^{\infty}(\overline{\Omega})$ *functions never negative in* $\overline{\Omega}$ *which represent classical solutions of the equation* $\mathrm{L}_1 u=0$ (*with* $\alpha=c^2$, $\beta=1$, $|x|=r$).

Furthermore, if $q=[(s+\alpha/\varepsilon)(s+\beta/\varepsilon)]^{\frac{1}{2}}$ and $\mathrm{H}(t)$ denotes the Heaviside function, the inversion formula obtained in [5] can be thus generalized.

LEMMA 2.2. *In the half-plane* $\mathscr{Re}(s)>\max \mathscr{Re}(-\alpha/\varepsilon, -\beta/\varepsilon)$ *the Laplace integral* $\mathscr{L}_t\,\mathrm{H}(t-r)\,\mathrm{F}_{\mu,\nu}$ *converges absolutely and one has*:

$$(2.5)\qquad \mathscr{L}_t\,\mathrm{H}(t-r)\,\mathrm{F}_{\mu,\nu}=2^{\mu+1}\,b^{-\nu}\,\frac{(s+\gamma-q)^{\nu}}{q\,(s+\delta+q)^{\mu+1}}\,e^{-rs\sqrt{\frac{\varepsilon s+\beta}{\varepsilon s+\alpha}}},$$

provided that both $\mathscr{Re}(\mu)$ *and* $\mathscr{Re}(\nu)$ *exceed* -1.

By means of this formula, and of other similar which are meaningful though $\mathscr{Re}(\mu)$ or $\mathscr{Re}(\nu)$ don't exceed -1, various Laplace transforms such as (0.2) can be inverted. In particular the explicit inverse $\mathscr{L}$-transforms of the $\hat{\mathrm{E}}_1$, $\hat{\mathrm{E}}_3$, $\hat{\mathrm{F}}_0$ functions defined in (1.8)-(1.9) can be deduced as follows. If $\alpha=c^2$ and $\beta=1$, then in (2.1) one has

$$(2.6)\qquad \gamma=\frac{1+c^2}{2\varepsilon}\quad,\quad b=\frac{1-c^2}{2\varepsilon}\quad,\quad \delta=\frac{c^2}{\varepsilon}.$$

If, for brevity, one puts

$$(2.7)\qquad \eta=(b/2)(t-r)\quad,\quad \xi=2\,[b\delta r\,(t-r)]^{\frac{1}{2}}\quad,\quad \omega=b\,(t^2-r^2)^{\frac{1}{2}},$$

$$(2.8)\qquad \mathrm{F}_3(r,t)=\varepsilon^{-1}\,e^{-\gamma t+\delta r}\times$$

$$\times\left[\mathrm{I}_0(\omega)+\int_0^1[4\eta v\,\mathrm{I}_0(\xi v)+\xi\mathrm{I}_1(\xi v)]\,\mathrm{I}_0(\omega\sqrt{1-v^2})\,e^{\eta v^2}\,dv\right]$$

$$(2.9)\qquad \mathrm{F}_1(r,t)=\int_r^t \mathrm{F}_3(z,t)\,dz\quad,\quad \mathrm{F}_0(r,t)=(\varepsilon\partial_t+c^2)\,\mathrm{F}_3,$$

accordingly [5] it is possible to prove that:

A) *The fundamental solution* E_3 *of* L_3 *operator* is

$$E_3(x, t) = (4\pi |x|)^{-1} H(t - |x|) F_3(|x|, t) \qquad x \in R^3 \tag{2.10}$$

where F_3 is the $C^\infty(\overline{\Omega})$ function defined in (2.8).

B) *The fundamental solution* E_1 *of* L_1 *operator* is

$$E_1(x, t) = (1/2) H(t - |x|) F_1(|x|, t) \qquad x \in R \tag{2.11}$$

where F_1 is the $C^\infty(\overline{\Omega})$ function defined in $(2.9)_1$.

C) *The kernel* E_0 *of the half-space problem* is the distribution

$$E_0(x, t) = e^{-bx} \delta(t - x) + H(t - x) F_0(x, t) \qquad x \in R^+ \tag{2.12}$$

where δ is the Dirac measure in R and F_0 is the $C^\infty(\overline{\Omega})$ function defined in $(2.9)_2$.

To state some maximum principles for the solution of boundary value problems related to L_3 and L_1 operators, we observe that by means of recurrence's formulae it can be proved

LEMMA 2.3. *The* F_j*'s* $(j = 0, 1, 3)$ *are* $C^\infty(\overline{\Omega})$ *functions never negative in* $\overline{\Omega}$. *Furthermore, everywhere in* $\overline{\Omega}$, *one has*:

$$\begin{gathered} \partial_t F_1 \geq 0 \quad , \quad \partial_r F_1 \leq 0 \quad , \quad (\varepsilon\partial_t + c^2) F_i \geq 0 \qquad (i = 1, 3) \\ \partial_r (\varepsilon\partial_t + c^2) F_1 \leq 0 \quad , \quad \partial_t (\varepsilon\partial_t + 1) F_1 \geq 0 \\ [\varepsilon(\partial_t^2 - \partial_r^2) + \partial_t] F_1 \geq 0 . \end{gathered} \tag{2.13}$$

Remark 2.1. The $F_{\mu,\nu}$'s are rapidly decreasing functions [5, 6]; then, rigorous estimates of various physical phenomena such as diffusion, asymptotic behaviour and singular perturbations can be obtained by means of the formulae analysed. Such questions will be dealt with successively (for an outline see [8]).

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$$\varepsilon\partial_t\,(\partial_t^2 - a_1^2\,\Delta) + \partial_t^2 - a_0^2\,\Delta\,,$$

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