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On the adjoint system to a very ample divisor on a surface and connected inequalities. Nota II

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Geometria algebrica. — *On the adjoint system to a very ample divisor on a surface and connected inequalities* (*). Nota II di ANTONIO LANTERI (**) e MARINO PALLESCHI (***), presentata (****) dal Corrisp. E. MARCHIONNA.

RIASSUNTO. — Si caratterizzano alcune classi di superfici in relazione all'indice di autointersezione dell'aggiunto ad un divisore molto ampio.

This Nota II is the second part of a work the first three sections of which appears in the same titled Nota I contained in the same tome of this review.

4. RATIONAL SURFACES RULED IN CONICS

Theorem 3.1 supplies a characterization of the surfaces ruled in conics. In this sec. such surfaces are more closely studied in the rational case. We have

THEOREM 4.1. *Let* $S \subset \mathbf{P}^n$ *be a regular surface with sectional genus* $g \geq 2$. *Then its general hyperplane section is a hyperelliptic curve if and only if* S *is a rational surface ruled in conics.*

Proof. The if part is immediate. Indeed consider the morphism $\pi : S \to \mathbf{P}^1$ whose fibres are conics. Then the restriction $\pi|_H : H \to \mathbf{P}^1$ is a morphism of degree two. To see the only if part notice that the map Φ_{K+H} is a morphism taking values in $\mathbf{P}^{p_g+g-1}$, in view of Remark 1.3 and formula (1.3). Put $\Sigma = \Phi_{K+H}(S)$; of course Σ cannot be a point, being $g \geq 2$. Now fix a generic point $x \in S$. By assumption, on any smooth hyperplane section H through x there is a point y which is the conjugate of x in the hyperelliptic involution of H. So, by adjunction, $\Phi_{K+H}(x) = \Phi_{K+H}(y)$ for any point y conjugate of x and that holds for $x \in S$ out of a Zariski closed subset. Hence Φ_{K+H} cannot be generically finite and then $\dim \Sigma \neq 2$. So $\dim \Sigma = 1$ and the map $\Phi_{K+H} : S \to \Sigma$ is a morphism in view of Remark 1.3. The Stein factorization

$$\begin{array}{ccc} S & \xrightarrow{\Phi_{K+H}} & \Sigma \\ & {\scriptstyle p}\searrow \quad \nearrow & \\ & C & \end{array}$$

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shows that $K + H$ is algebraically equivalent to a finite sum of fibres of p; hence $(K + H)^2 = 0$. Now, as $g \geq 2$, Theorem 3.1 says that S is ruled in conics; finally S is rational as $q = 0$.

A classical result due to Enriques [8] claims that a surface $S \subset \mathbf{P}^n$ whose general hyperplane section is a hyperelliptic curve is either a scroll or a rational surface. This fact together with the characterization given in Theorem 4.1 has the following

COROLLARY 4.1. *Let $S \subset \mathbf{P}^n$ be a surface with hyperelliptic hyperplane sections of genus $g \geq 2$. Then S is either a scroll or a rational surface ruled in conics.*

Remark 4.1. ([11], p. 434). For a surface $S \subset \mathbf{P}^4$ of degree d the following formula holds

$$d^2 - 10\,d + 12\,\chi(\mathcal{O}_S) = 2\,K^2 + 5\,HK\,. \tag{4.1}$$

PROPOSITION 4.1. *Let $S \subset \mathbf{P}^4$ be a surface whose general hyperplane section is a hyperelliptic curve of genus $g \geq 2$. Then $g = 2$ and S is a quintic rational surface represented on $\mathbf{P}^2$ by a linear system of nodal quartics $\delta = |C_4 - 2\,p_1 - p_2 - \cdots - p_8|$* (4) *the points p_i $(i = 1, \cdots, 8)$ being in general position.*

Proof. It is known (see [13]) that in $\mathbf{P}^4$ there are no scrolls of sectional genus $g \geq 2$. So, in view of Corollary 4.1, S can only be a rational surface ruled in conics. By Corollary 3.1, II and formula (4.1) one gets $g = \frac{1}{2}(d^2 - 7\,d + 14)$. So Castelnuovo's inequality (see [11], p. 351) implies $4 \leq d \leq 6$, but, as $g \geq 2$, it can only be $d \geq 5$. On the other hand, if $d = 6$, one obtains $g = 4$ and then (see [9], p. 247) the general hyperplane section of S would be a canonical curve. which is absurd. Then $d = 5$, $g = 2$ and we are done (e.g. see [14], Th. 5.1),

5. A SECOND INEQUALITY AND A CHARACTERIZATION OF THE RATIONAL SURFACES RULED IN CUBICS

We need to point out some other properties of $|K + H|$.

Remark 5.1. Suppose S is neither a scroll nor ruled in conics. If $g \geq 3$, then Φ_{K+H} is a morphism and $\Sigma = \Phi_{K+H}(S)$ has dimension two. First of all $\dim \Sigma \geq 1$ in view of Proposition 3.1 and Lemma 2.4. Secondly Φ_{K+H} is a morphism by [25], Propositions 2.0.1 and 1.5. By absurd, suppose $\dim \Sigma = 1$. The same argument on the Stein factorization, used before to conclude the proof of Theorem 4.1, shows that $(K + H)^2 = 0$. This is absurd, by Theorem 3.1.

(4) Let $p_1, \cdots, p_r$ be distinct (or infinitely near) points of $\mathbf{P}^2$. As usual (see [11], p. 395) the symbol $|C_m - s_1\,p_1 - \cdots - s_r\,p_r|$ denotes the linear system of the plane curves of order m having a point of multiplicity s_i at p_i $(i = 1, \cdots, r)$.

Remark 5.2. Let $\Phi : S \to S_0$ be a birational morphism taking values in a smooth surface S_0. If Φ contracts only r exceptional curves of the first kind, then Φ factorizes via r simultaneous blowings-up (i.e. the r blowings-up have distinct centers on S_0). Moreover, if $\Phi_{K+H} : S \to \Sigma$ is a birational morphism, then

α) Σ has no isolated singularity;

β) if Σ is smooth then Φ_{K+H} factorizes via simultaneous blowings-up. Suppose Φ factorizes via the blowings-up

$$S = S_r \xrightarrow{\sigma_{r-1}} S_{r-1} \xrightarrow{\sigma_{r-2}} \cdots \xrightarrow{\sigma_{i+1}} S_{i+1} \xrightarrow{\sigma_i} S_i \xrightarrow{\sigma_{i-1}} S_{i-1} \longrightarrow \cdots \xrightarrow{\sigma_0} S_0$$

and suppose, by absurd, the blowing-up σ_i has its center p on a curve Γ contracted by $\sigma_{i-1} \circ \sigma_{i-2} \circ \cdots \circ \sigma_0$. Then $\sigma_i^* \Gamma = \sigma_i^{-1}(\Gamma) + E$, E being the exceptional curve of the first kind corresponding to p. Then

$$-1 \geq \Gamma^2 = (\sigma_i^* \Gamma)^2 = (\sigma_i^{-1}(\Gamma))^2 + \varepsilon ,$$

with $\varepsilon \geq 1$. Then the proper transform of Γ in S_{i+1}, and then in S, is not an exceptional curve of the first kind. Now suppose $\Gamma \subset S$ is an irreducible curve contracted by Φ_{K+H}. As $\Gamma(K+H) = 0$ we see $\Gamma K = -\Gamma H < 0$. On the other hand, as $\Gamma^2 < 0$ (see [15], p. 6), we must have by genus formula $-2 \leq 2g(\Gamma) - 2 = \Gamma^2 + \Gamma K \leq -2$. We thus see that Γ is an exceptional curve of the first kind. This proves α). Statement β) follows from the first part.

Let $S \subset \mathbf{P}^n$ be a ruled surface which is neither a scroll nor ruled in conics; we call S *ruled in cubics* if its fibres have degree three.

LEMMA 5.1. *Let $S \subset \mathbf{P}^n$ be a surface of sectional genus g ruled in cubics. Then*

$$(K+H)^2 = g + q - 2 . \tag{5.1}$$

Proof. Let S_0 be a geometrically ruled surface of irregularity q and consider a fundamental section C_0 and a fibre f of its. If C is a smooth three-secant curve of S_0 (i.e. $Cf = 3$), for a suitable integer m one has $C \equiv 3C_0 + mf$. As $K_{S_0} \equiv -2C_0 + (2q - 2 - e)f$, by (1.5), a straightforward calculation gives

$$(K_{S_0} + C)^2 = g(C) + q - 2 . \tag{5.2}$$

Consider now the surface S ruled in cubics. If $S = S_0$ is geometrically ruled in cubics its general hyperplane section H is a smooth three-secant curve of S_0 and (5.2) becomes (5.1). Otherwise, the singular fibres of S are reducible and each reducible fibre F of S is one of the following:

a) $F = \Gamma + L$, with $\Gamma H = 2$, $LH = 1$, $\Gamma^2 = L^2 = -1$ and $\Gamma L = 1$;

b) $F = L_1 + L_2 + L_3$, with $L_i H = 1$ $(i = 1, 2, 3)$, $L_1^2 = L_3^2 = -1$, $L_2^2 = -2$ and $L_1 L_2 = L_3 L_2 = 1$, $L_1 L_3 = 0$.

This follows immediately from genus formula, the rationality of the general fibre of S and the fact that $F^2 = 0$. Now consider the morphism $\eta : S \to S_0$ blowing-down the exceptional lines L's on each fibre *a*) and L_1 and L_3 on each fibre *b*). By Castelnuovo's criterion (see [24], p. 36) S_0 is a smooth surface; moreover it is immediate to see that S_0 is geometrically ruled. Now notice that a general hyperplane section H of S is the proper transform via η of a smooth three-secant curve C on S_0. Indeed, call $p_1, \cdots, p_r$ the points of S_0 to which η contracts the quoted exceptional lines; the curve C, image of H, is a curve through $p_1, \cdots, p_r$ and it is smooth, since $H\eta^{-1}(p_i) = 1$. As

$$\eta^* C = H + \sum_{i=1}^{r} \eta^{-1}(p_i) \qquad \text{and} \quad \eta^* f = F\,,$$

for a fibre f of S_0 outside of $p_1, \cdots, p_r$, there follows

$$Cf = \eta^* C\, \eta^* f = HF + \sum_{i=1}^{r} \eta^{-1}(p_i)\, F = HF = 3\,.$$

But, as it is known, $K = \eta^* K_{S_0} + \sum_{i=1}^{r} \eta^{-1}(p_i)$, and so $\eta^*(K_{S_0} + C) = K + H$. Then $(K+H)^2 = (K_{S_0} + C)^2$ and since $g(C) = g(H) = g$, (5.2) gives (5.1).

THEOREM 5.1. *Suppose* $S \subset \mathbf{P}^n$ *is a surface with sectional genus* $g \geq 3$. *If* S *is neither a scroll nor ruled in conics, then*

$$(K+H)^2 \geq p_g + g - q - 2\,, \tag{5.3}$$

and equality holds if and only if S *is one of the following rational surfaces*:

i) *a Bordiga surface i.e. the image of* $\mathbf{P}^2$ *via the rational map associated to a linear system* $|C_4 - p_1 - \cdots - p_r|$ *of quartics through* r $(0 \leq r \leq 10)$ *distinct points* p_i *in general position*;

ii) *the image of* $\mathbf{P}^2$ *via the rational map associated to a linear system* $|C_5 - p_1 - \cdots - p_s|$ *of quintics through* s $(0 \leq s \leq 15)$ *distinct points in general position*;

iii) *a rational surface ruled in cubics.*

Proof. First of all $\Phi_{K+H} : S \to \Sigma$ is a morphism and $\dim \Sigma = 2$, in view of Remark 5.1. Moreover the (possibly singular) surface Σ is contained in $\mathbf{P}^{p_g+g-q-1}$ by (1.3) and then it has degree $\geq p_g + g - q - 2$. It thus follows the inequality

$$(K+H)^2 = \deg \Phi_{K+H} \deg \Sigma \geq p_g + g - q - 2\,; \tag{5.4}$$

so (5.3) is established.

Now suppose equality holds in (5.3). Then $\deg \Phi_{K+H} = 1$ and $\deg \Sigma = p_g + g - q - 2$, by (5.4). So Φ_{K+H} is a birational morphism and Σ falls in one of the following cases (see [21], p. 607):

1) $\Sigma = \mathbf{P}^2$,
2) Σ is the Veronese surface,
3) Σ is a rational scroll,
4) Σ is a cone over a rational normal curve.

First of all note that case 4) does not occur in view of Remark 5.2, α. In any case the surface S is rational and then $\deg \Sigma = g - 2$ and $\Sigma \subset \mathbf{P}^{g-1}$. In case 1), it is $g = 3$ and Remark 5.2, β shows that Φ_{K+H} factorizes via r simultaneous blowings-up. As $H(K + H) = 4$, S is as in *i*). Really as the linear system $|C_4 - p_1 - \cdots - p_r|$ embeds S in $\mathbf{P}^{14-r}$, it must be, of course, $r \leq 10$. In case 2) arguments similar to the previous ones show that $g = 6$, $H(K + H) = 10$ and that S is as in *ii*). In case 3) call f a fibre of the scroll Σ and consider the proper transform $C = \Phi_{K+H}^{-1}(f)$. So $C^2 = 0$. Moreover $g(C) = 0$ as Φ_{K+H} is birational and then $CK = -2$. Since $1 = \deg f = C(K + H)$, it thus follows $CH = 3$ and S is ruled in cubics. Conversely, in cases *i*) and *ii*) a straightforward computation shows equality in (5.3). In case *iii*) equality follows from Lemma 5.1.

6. On projective surfaces of low sectional genus

The classification of surfaces with a given sectional genus g is a quite classical subject in Algebraic Geometry. This was treaten for low values of g by many geometers; the most important contributions we know are due to Noether, Picard [17], Castelnuovo [3], [4], [5], Enriques [8], Scorza [22], and Roth [18], [19], [20]. Some results proven in previous sections apply specifically to the study of surfaces of low sectional genus. For giving an example here we restate some of the known results for $g \leq 4$ supplying a unitary proof of them; by the way we point out some facts in cases $g = 3$ and $g = 4$.

As we shall see in a moment the most of surfaces with low g are ruled. Hence it is convenient for the sequel to point out the first inequality of sec. 3 for ruled surfaces.

From now on $S \subset \mathbf{P}^n$ will be a surface of degree d, H its general hyperplane section and $g = g(H)$.

PROPOSITION 6.1. *Suppose* S *is a linearly normal ruled surface. If* S *is neither a scroll nor the Del Pezzo surface of degree* $d = 9$, *one has*

$$d \leq 4g + 4 - 8q \quad (6.1)$$

(5).

(5) Compare (6.1) with the inequalities proven by Hartshorne (see [12], pp. 115–120) for the self-interesection of a curve of positive genus on a ruled surface.

Moreover equality holds if and only if S *is either*

i) geometrically ruled in conics,

ii) the Veronese surface, or

iii) the Bordiga surface of degree $d = 16$ *(i.e.* S *is* $\mathbf{P}^2$ *embedded by the complete linear system of all quartics).*

Proof. Suppose $S \simeq \mathbf{P}^2$. Then S is $\mathbf{P}^2$ embedded by the complete linear system of curves of degree m and $g = \frac{1}{2}(m-1)(m-2)$, $d = m^2$. So (6.1) is fulfilled unless $m = 3$ and equality holds if and only if either $m = 2$ or $m = 4$, i.e. in cases *ii*) and *iii*). Now suppose $S \not\simeq \mathbf{P}^2$; then Remark 1.5 implies $K^2 \leq 8(1-q)$ and so (3.5) supplies (6.1). Equality holds if and only if equality holds in (3.5) and simultaneously S is geometrically ruled, i.e. in case *i*) by Corollary 3.1.

By Proposition 6.1, recalling Proposition 3.1, Remarks 1.1, 3.2 and Theorem 4.1 we get immediately the classical results when $g \leq 2$.

THEOREM 6.1 (Picard-Castelnuovo-Del Pezzo). *If* S *is a linearly normal surface of sectional genus* $g \leq 2$, *then*

i) S *is either* $\mathbf{P}^2$, *the Veronese surface or a rational scroll, if* $g = 0$;

ii) S *is either a Del Pezzo surface or an elliptic scroll, if* $g = 1$;

iii) S *is either a rational surface of degree* d $(5 \leq d \leq 12)$ *ruled in conics (with* $\delta = 12 - d$ *singular fibres) or a scroll, if* $g = 2$.

To analyze case $g = 3$ we need the following lemma the proof of which makes also use of rather classical arguments (e.g. see [6], pp. 149–150).

LEMMA 6.1. *Let* $S \subset \mathbf{P}^{d-2}$ *be a surface of degree* d *with* $g = 3$. *Then* S *is rational.*

Proof. As $d \geq 6$ by Castelnuovo's inequality (see [11], p. 351), one has $d > 2g - 2$ and then S is ruled in view of Remark 1.4. By projecting S from $d - 5$ points of itself in a $\mathbf{P}^3$ we obtain a singular surface $S' \subset \mathbf{P}^3$ of degree five. Consider the minimal desingularization $\eta : \Sigma \to S'$ of S' and the divisors H' and Δ' which are the inverse images via η of a hyperplane section of S' and of its double curve Δ respectively. Then (e.g. see [9], p. 627).

$$K_\Sigma \equiv H' - \Delta' . \tag{6.2}$$

Now, since Σ is birational to the ruled surface S, we have $p_g(\Sigma) = 0$, and so (1.3) reads

$$h^0(K_\Sigma + H') = 3 - q . \tag{6.3}$$

Afterwards consider in $\mathbf{P}^3$ a general line l skew with Δ and two points q_1, q_2 on l. A plane Π through l cuts out on S' a quintic with three double points

p_1, p_2, p_3. Obviously these points are not collinear and none of them lies on l; hence there exists a unique irreducible conic on Π through the five points p_1, p_2, p_3, q_1, q_2. When Π varies in the pencil with base l this conic generates a quadric surface Q containing Δ. Denote by Q′ the divisor on Σ inverse image via η of $Q \cap S'$. One has $Q' \in |2H' - \Delta'|$, hence $Q' \in |K_\Sigma + H'|$ in view of (6.2). Now when q_1 and q_2 vary on l, the quadric Q (and the Q′) varies in a net. So (6.3) involves $q = 0$; thus S is rational.

THEOREM 6.2 (see also Castelnuovo [5]). *Let* S *be a surface of sectional genus* $g = 3$. *Then* S *is either*

i) *a rational surface of degree* d *with* $6 \le d \le 16$,

ii) *a linearly normal surface of degree* $d = 8$ *in* $\mathbf{P}^5$ *geometrically ruled in conics over an elliptic curve and with invariant* $e = -1$,

iii) *a scroll* (*with* $q = 3$), *or*

iv) *a quartic surface in* $\mathbf{P}^3$.

Proof. If $S \subset \mathbf{P}^3$ it must be $d = 4$ and we are in case iv). Otherwise $d \ge 6$ by Castelnuovo's inequality (see [11], p. 351). In this case S is ruled by Remark 1.4 and it has irregularity $q \le 3$ in view of Remark 1.1. If $q = 3$ we are in case iii) by Proposition 3.1. Suppose $q \le 2$; as $d \ge 6$, (6.1) implies $q \le 1$. If $q = 0$ then S is rational and (6.1) again supplies $d \le 16$; so we are in case i). It remains only to show that if $q = 1$ we are in case ii). First of all, it must be $d \le 8$, by (6.1). Moreover the exact sequence (1.4) gives $h^0(H) = d - 2 + h^1(H)$ and $h^1(H) = \varepsilon \le 1$, by Remark 1.4. So $S' = \Phi_H(S)$ is a surface of degree d in $\mathbf{P}^{d-3+\varepsilon}$ with sectional genus three. Thus, being $q = q(S') = 1$, Lemma 6.1 implies $\varepsilon = 0$ i.e. $S' \subset \mathbf{P}^{d-3}$. First of all it is $d \ne 6$, S′ being irregular. By absurd suppose $d = 7$; then formula (4.1) shows $K^2 = -3$ whilst $K^2 \ge -1$, by (3.5). There thus follows $d = 8$ and then $S' \subset \mathbf{P}^5$. On the other hand S, cannot be a projection of S′, otherwise S should be singular, since S′ is not the Veronese surface (6) (see [23]). So $S = S'$ is a linearly normal elliptic ruled surface of degree $d = 8$ in $\mathbf{P}^5$. Moreover, equality holding in (6.1), Proposition 6.1 says that S is geometrically ruled in conics. Finally to determine the invariant e of S consider a fibre F_1 and let Π be a hyperplane containing the plane $\langle F_1 \rangle$. The corresponding hyperplane section of S is $H_1 = F_1 + \Gamma$, where $\Gamma \equiv 2C_0 + (m-1)F$, in view of (1.7). As C_0 is an elliptic curve, we must have $\deg C_0 = C_0 H \ge 3$; so we get

$$6 = (H_1 - F_1)H = \Gamma H = (2C_0 + (m-1)F)H \ge 6 + 2(m-1)$$

and then $m \le 1$. Now $8 = H^2 = 4C_0^2 + 4m$; hence $e = -C_0^2 = -2 + m$. As $e \ge -1$ (see [11], p. 377), one gets $m \ge 1$ and so $m = 1$ and $e = -1$.

(6) Suppose S is a projection of S′. Then the center of the projection must be outside S′, the two surfaces having the same degree.

By the way it is worth mentioning that the projective configuration occurring in case ii) for the elliptic system of conics on S had been deeply described by Scorza in [22]. Moreover, as far as an explicit description (including the plane models) of the rational surfaces occurring in case i) is concerned, see [7], pp. 489–490.

Now we are able to point out a fact in [14].

COROLLARY 6.1. *Let* $S \subset \mathbf{P}^4$ *be a surface of degree* $d = 6$. *Then* S *is either* 1) *a Bordiga surface* (i.e. the image of $\mathbf{P}^2$ via the rational map associated to a linear system of quartics through ten points in general position), *or* 2) *a complete intersection of a quadric and a cubic form.*

Proof. If we are not in case 2), S is a ruled surface and $g = 3$ (see [14], sec. 6). Lemma 6.1 shows that S is rational and formula (4.1) implies $K^2 = -1$. Then S is not ruled in conics, by Corollary 3.1, II. Moreover $HK = 2g - 2 - d$ and so $(K + H)^2 = 1$. As $h^0(K + H) = g = 3$, it turns out that $\Phi_{K+H} : S \to \mathbf{P}^2$ is a birational morphism and it factorizes by means of ten simultaneous blowings-up, in view of Remark 5.1, β. As $H(K + H) = 4$, Φ_{K+H} relates $|H|$ to a linear system of plane quartics with ten simple base point.

THEOREM 6.3. *Let* S *be a surface of degree d with sectional genus* $g = 4$. *Then* S *is either*

i) *a rational surface with* $7 \leq d \leq 20$,

ii) *an elliptic ruled surface with* $8 \leq d \leq 12$,

iii) *a scroll* (*with* $q = 4$), *or*

iv) *the complete intersection of a quadric and a cubic form of* $\mathbf{P}^4$.

Proof. First of all $d \geq 6$ by Castelnuovo's inequality. Suppose $d = 6$; as $d = 6 = 2g - 2$, on a general hyperplane section H, the characteristic linear series $||H|\cdot H|$ has dimension $h^0(H|_H) \leq 4$, equality holding if and only if $H|_H \equiv K_H$ (e.g. see [10], p. 111). Were it $h^0(H|_H) < 4$ it would be $S \subset \mathbf{P}^3$, as we can see by (1.4) and so H would be a (smooth) plane curve of genus 4: absurd. Henceforth H is a canonical curve of genus $g = 4$ and so it is the complete intersection of a quadric and a cubic form of $\mathbf{P}^3$; then we are in case iv) (e.g. see [11], p. 276). Now suppose $d \geq 7$. By Remarks 1.4 and 1.1, S is ruled and $q \leq 4$. If $q = 4$, we are in case iii) by Propositions 3.1; otherwise formula (6.1) shows $q \leq 1$ as $d \geq 7$. If $q = 0$, S is rational and (6.1) once again gives $d \leq 20$ and so we are in case i). Suppose now $q = 1$; hence S is an elliptic ruled surface and (6.1) supplies $d \leq 12$. It remains to show that $d \geq 8$. By absurd, suppose $d = 7$; since S cannot be contained in $\mathbf{P}^3$, the exact sequence (1.4) gives $h^0(H) = 5$, namely $S \subset \mathbf{P}^4$. Thus formula (4.1) supplies $K^2 = -8$. On the other hand it must be $K^2 \geq -5$ by (3.5).

Of course, by Proposition 6.1, the surfaces of degree $d = 20$ in i) are geometrically ruled in conics. Note also that in view of Theorem 5.1 the surfaces of degree $d = 19$ in i) are forced to be ruled in conics.

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