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Complements of analytic subvarieties and q-complete spaces

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Geometria. — *Complements of analytic subvarieties and* q-*complete spaces*. Nota di EDOARDO BALLICO (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si dimostra che il complementare $X \setminus Y$ di un sottospazio analitico chiuso localmente intersezione completa di codimensione q di una varietà di Stein è q-completo.

INTRODUCTION

We want to study the problem of the pseudoconvexity of the complement of a closed analytic subvariety Y in a Stein manifold. It is well-known that this problem is completely solved if the subvariety Y has codimension 1 in X. In the general case the problem was studied by G. Sorani and V. Villani in [8], prop. 1. Their proof did not give the best result because with their method one direction of positivity for the Levi form of an exaustion function is lost. It seems to us that their proof can be strenghtened and the best result obtained, because the lost direction of positivity is a direction more or less along the gradient of the exaustion function.

But in the meantime Fritzsche in [4] and [5] gave very good results of convexity for the complement of subvariety with positive normal bundle in a compact space. Fritzsche's method is very powerful and we show that using it we can solve our problem.

We use Fritzsche's notation as far as possible. In particular, as in [1], in this paper 1-complete space is equivalent to Stein space and 1-convex function means strictly plurisubharmonic function.

We consider the following situation. X is a complex manifold and Y is a locally complete intersection subspace of codimension q of X.

In the first paragraph we show that if X is a Stein manifold, the $X \setminus Y$ is q-complete. The proof uses the positivity of any vector bundle on a Stein space and an extensione of Satz 4.10 in [4] about q-concave linear spaces from the case of a compact space to the general case.

In the second paragraph we study the same problem when X is not a Stein space. Our first result concerns a very particular case but does not assume an ambient Stein space. Theorem 2 is the last result of the second paragraph. It says that the complement of a q-codimensional locally complete intersection Y in a r-complete manifold X is $(q + r - 1)$-complete if X is an open subset

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of a Stein space and the normal bundle of Y in X is generated by global sections. In Theorem 1 and 2 the ambient variety X is always non singular, although the propositions used for the proofs are true even for non reduced complex spaces. The hypothesis of non singularity can be relaxed. For example it is sufficient to assume that the singular locus of X does not intersect Y.

§ 1. We use the notation of [4]. For us the word "differentiable" means "of class C^∞". Let X be a complex space. A differentiable function f on X is q-convex if for any $x \in X$ the Levi form $\mathrm{Lev}_x(f)$ of f at the point x has at most $q-1$ non-positive eigenvalues. In particular a differentiable function f on X is 1-convex if and only if it is strictly plurisubharmonic.

A complex space X is q-complete if there exists a differentiable, positive function f on X which is q-convex and exahustive, i.e. such that the sets $\{x \in X : f(x) < c\}$ are relatively compact for each positive real number c.

A complex linear fiber space (or a linear space) is a quintuple $(E, \pi, \alpha, \beta, \omega)$ where $\pi : E \to X$, $\alpha : E \times_X E \to E$, $\beta : \mathbf{C} \times E \to E$ and $\omega_E : X \to E$ are holomorphic maps such that the natural diagrams are commutative (here α is the addition, β is the scalar multiplication, ω_E is the zero section). Therefore the projection $\pi : E \to X$ is a family of vector space parametrized by X. There is a canonical antiequivalence between the category of linear spaces on a fixed complex space X and the category of analytic coherent sheaves on X. For any analytic coherent sheaf $\mathscr{F}$ on X, we write $\mathrm{Lin}(\mathscr{F})$ for the associated linear space. By definition the tangent bundle T(X) to X is the linear space $\mathrm{Lin}(\Omega_X)$ associated to the cotangent sheaf Ω_X. For more details on the definition and che elementary properties of a linear space, the reader can see [4], [5] or [3].

Let $\pi : E \to X$ be a linear space on a complex space. For each $v \in E$, there is a canonical exact sequence

$$0 \to E_x \to T_{E,v} \xrightarrow{T_v(\pi)} T_{X,x}$$

where $x : \pi(v)$. The image of E_x in $T_{E,v}$ is the relative tangent bundle $T(E/X)_v$ of π at v. We put $B_v(E) := \mathrm{Im}\, T_v(\pi) \subset T_{X,x}$. For each pseudotrivialization $\varphi : E_{|\pi^{-1}(U)} \hookrightarrow U \times \mathbf{C}^q$ of E around x, we obtain a splitting $T_{E,v} \cong T(E/X) \oplus B_v^{(\varphi)}(E)$.

DEFINITION 1. Let X be a complex space, $\pi : E \to X$ be a linear space. E is said r-positive, if there exists a scalar product h on E such that for each $x \in X$, for each $v \in E_x \setminus \{0\}$ and for any pseudotrivialization φ around the point x, the Levi form $\mathrm{Lev}_v(h)$ has a most $r-1$ non-negative eigenvalues on the vector space $B_v^{(\varphi)}(E)$.

Let $\pi : E \to X$ be a linear space on a complex space and $\tau_E : E \to T(E)$ be the canonical embedding. For each open set $M \subset E$ and for each differentiable function f on M, we put $Ff(v) := (df)_v(\tilde{\tau}_E(v))$ for $v \in M$. In this way Fritzsche defines a differentiable function Ff on M: the differential of f along the fibers of E.

DEFINITION 2. Let X be a complex space and $\pi : E \to X$ a linear space. E is said q-concave if there exists an open neighborhood U of the zero-section $\omega_E(X)$ such that:

a) for any $x \in X$, $U \cap E_x$ is a relatively compact and starred neighborhood of the zero in E_x;

b) there exists an open neighborhood V of the boundary ∂U of U and a q-convex function s on V with $V \cap U = \{u \in V : s(v) > 0\}$ and with $Fs(v) \neq 0$ for each $v \in \partial U$.

PROPOSITION 1. *Let* X *be a Stein space and* E *be a linear space on* X. *Then* E *is* 1*-positive.*

The proposition above was proved by S. Nakano in [6] when X is a Stein manifold and E is a vector bundle. The proof given in [7] can be easily extended to the general case.

PROPOSITION 2. *Let* X *be a complex space and* $\pi : E \to X$ *be a linear space. We put* $q := \sup_{x \in X} \dim_{\mathbf{C}}(E_x)$. *If* E *is r-positive, then* E *is* $(q + r - 1)$*-concave.*

Proof. We extend to the general case the proof given in [4], Satz 4.10, when X is a compact space. Let h be a scalar product on E such that h defines the r-positivity of E. Let v_0 be a point of E. We put $s := h - h(v_0)$ and $\rho_u := se^{us}$, where u is a positive differentiable function on X.

We put $W := \{v \in E : \rho_u(v) < 0\}$. W is a neighborhood of the zero-section $\omega_E(X)$ of E and for each relatively compact open subset U of X, $\pi^{-1}(U)$ is relatively compact in E. In the proof of [4], Satz 4.10, u is a sufficiently big costant. For each point $v \in \partial W$ the differential and the Levi form of ρ_u and $\rho_{(u)v}$ are the same because every term in which a derivative of u appears is multiplied by $s(v) = 0$. Fritzsche proved that we have $Fs(v) \neq 0$ for each $v \in \partial W$ and that for each point $x \in X$ there exists a positive constant $c_0(x)$ such that if $c \geq c_0(x)$, then ρ_c is $(q + r - 1)$-convex at every point of $\partial W \cap E_x$.

In our case it is sufficient to take for u a differentiable function with $u(x) \geq c_0(x)$ for each $x \in X$. In fact for such a function ρ_u is $(q + r - 1)$-convex at every point of ∂W and therefore also in a neighborhood of ∂W.

Thus ρ_u, W and V satisfy the conditions *a*) and *b*) of Definition 2 and Proposition 2 is proved. $\square$

Let X be a complex space and Y be a closed, analytic subspace of X. Y is defined by a coherent sheaf of ideals $\mathcal{T}$. The normal bundle of Y in X is by definition the linear space on Y associated to the coheren $\mathcal{O}_Y$-module $\mathcal{T}/\mathcal{T}^2_{|Y}$.

As a corollary of the results proved by Fritzsche in [4] we obtain the following theorem.

THEOREM 1. *Let* X *be a Stein manifold and* Y *be a closed analytic subspace of* X. *Let* N *be the normal bundle of* Y *in* X. *We put* $q := \sup_{X \in x} \dim(N_x)$. *Then* $X \setminus Y$ *is a q-complete complex manifold.*

Proof. From propositions 1 and 2 it follows that N is a q-concave linear space. From [4], Theorem 5.4, the main theorem of the cited paper by Fritzsche, it follows that there exists a differentiable function $f : X \setminus Y \to \mathbf{R}_+$ and an open neighborhood M of Y in X such that:

a) for each $x \in M \setminus Y$, the Levi form of f $\mathrm{Lev}_x(f)$ has at most $q-1$ eigenvalues ≤ 0;

b) if $x_0 \in Y$ and $\{x_u\}$ is a sequence of points in $M \setminus Y$ which converges to x_0, then the sequence $\{f(x_u)\}$ is unbounded.

Let p be a positive, 1-convex differentiable function on X such that for every $c \in \mathbf{R}_+$ the set $X_c := \{x \in X : p(x) < c\}$ is relatively compact in X. Let $M' \subset M$ be a neighborhood of Y with $\partial M' \subset M$. For each compact subset K of X, the Levi form of f $\mathrm{Lev}(f)$ has in $K \cap (X \setminus M')$ bounded eigenvalues and in $K \cap (M \setminus Y)$ at most $q-1$ negative eigenvalues. With a standard technique used in [2] and in [9] (for example it is sufficient to modify the statement of Lemma 1 in [9]) we obtain a differentiable, increasing function $\psi : \mathbf{R} \to \mathbf{R}$, with $\psi(0) \geq 0$, $\lim_{t \to +\infty} \psi(t) = +\infty$ and such that the function $\psi(p) + f$ is q-convex in $X \setminus Y$. Since for any $c \in \mathbf{R}$ the set $\{x \in X \setminus Y : \psi(p(x)) + f(x) < c\}$ is relatively compact in $X \setminus Y$, $X \setminus Y$ is q-complete. □

COROLLARY. *Let* X *be a Stein manifold and* Y *be a closed analytic subspace of* X. *If* Y *is a locally complete intersection of codimension* q *in* X, *then* $X \setminus Y$ *is a* q*-complete complex manifold.*

This problem was studied by G. Sorani and V. Villani in [8]. In their terminology a q-complete space is a $(q+1)$-complete space in our terminology. Therefore the Proposition 1 of [8], which in our terminology says more or less that, if Y is non singular, then $X \setminus Y$ is $(q+1)$-complete, is weaker than our corollary above. It is easy to see that the q-completeness of $X \setminus Y$ is the best result. If Y is not a locally complete intersection in X, then the corollary above is not true. G. Sorani and V. Villani gave an example in which the corollary above is not true. In this example there exists a point y of Y is which we have $\mathrm{embdim}_y(Y) = \dim_y X$. They take $X = \mathbf{C}^{2n}$, $n > 1$, with coordinate $z_1, \cdots, z_{2n}$ and Y the union of the linear subspace definined by the equations $z_1 = \cdots = z_n = 0$ and of the linear subspace defined by the equations $z_{n+1} = \cdots = z_{2n} = 0$. They prove with a Mayer-Vietoris exact sequence that we have $H^{2n-1}(X \setminus Y, \mathcal{O}_{X \setminus Y}) \neq 0$ and in particular $X \setminus Y$ is not $(2n-1)$-complete. In theorem 1 and in its corollary the condition of nonsingularity of X can be weakened. For example the same proof applies if the singular locus of X does not intersect Y.

§ 2. We want to prove with a direct calculation the q-concavity of some linear space. Our purpose is to obtain results similar to those of the preceding paragraph when X is not a Stein space.

PROPOSITION 3. *Let* X *be a complex space with a strictly plurisubharmonic and upper bounded differentiable function. Let* E *be a trivial vector bundle of rank* q *on* X. *Then* E *is a* q*-concave vector bundle.*

Proof. Let u be a strictly plurisubharmonic function on X with $u \leq -1$. The total space of E is $X \times \mathbf{C}^q$. Let $\pi : E \to X$ be the projection. Let $z = (z_1, \cdots, z_q)$ be the coordinates on $\mathbf{C}^q$. The function $s : E \setminus \omega_E(X) \to \mathbf{R}$, given by $s(x, z) = 1 \Big/ \left(\sum_{i=1}^{q} z_i \bar{z}_i\right) + u(x)$, defines the q-concavity of X.

Let U be defined by $U : \{(x, z) \in E : s(x, z) > 0\} \cup \omega_E(X)$. For each $x \in X$, $U \cap E_x$ is relatively compact in E_x and it is a starred neighborhood of zero in E_x. Furthermore s is q-convex. We take $(x, z) \in E \setminus \omega_E(X)$. We put $\rho = \sum_{i=1}^{q} z_i \bar{z}_i$.

Then we have

$$\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} s = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \frac{1}{\rho} = \frac{\partial}{\partial z_\alpha} \left(\frac{-z_\beta}{\rho}\right) = \frac{-\delta_{\alpha\beta}\rho + 2\bar{z}_\alpha z_\beta}{\rho^3}$$

and therefore we have

$$\mathrm{Lev}_{(x,z)}(s) = \begin{pmatrix} \mathrm{Lev}_{(x)}(u) & 0 \\ 0 & \dfrac{-\delta_{\alpha\beta}\rho + 2\bar{z}_\alpha z_\beta}{\rho^3} \end{pmatrix}.$$

Let L be the subspace $T(X)_x$ tangent to X at the point x and embedded in the tangent space $T(E)_{(x,z)}$ of E at the point (x, z). Obviously we have $T(E)_{(x,z)} \cong L \times \mathbf{C}^q$ and $\mathrm{Lev}_{(x,z)}(s)_{|L} \cong \mathrm{Lev}_{(x)}(u)$ is positive definite. Let H be the linear subspace of $T(E)_{(x,z)}$ generated by L and the point $(0, \cdots, 0, z_1, \cdots, z_q)$. H has codimension $q - 1$ in $T(E)_{(x,z)}$ because $(x, z) \notin \omega_E(X)$ i.e. we have $z \neq 0$.

$\mathrm{Lev}_{(x,z)}(E)_{|H}$ is definite positive. In fact we have

$$(0, \cdots, 0, \bar{z}_1, \cdots, \bar{z}_q) \begin{pmatrix} \mathrm{Lev}_{(x)}(u) & 0 \\ 0 & \dfrac{-\delta_{\alpha\beta}\rho + 2\bar{z}_\alpha z_\beta}{\rho^3} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ z_1 \\ \vdots \\ z_q \end{pmatrix} =$$

$$= \frac{-\left(\sum_\alpha z_\alpha \bar{z}_\alpha\right)\rho + 2\sum_{\alpha\beta} z_\alpha \bar{z}_\alpha z_\beta \bar{z}_\beta}{\rho^3} = +\frac{1}{\rho} > 0. \quad \square$$

As an application of the proposition above we obtain the following result.

PROPOSITION 4. Let X be a r-complete manifold and Y be a closed analytic subspace of X. Suppose that Y is a locally complete intersection of codimension q with trivial normal bundle. Suppose that there exists a differentiable function on Y which is strictly plurisubharmonic and upper bounded.

Then $X \setminus Y$ is $(r + q - 1)$-complete.

Proof. From proposition 3 it follows that the normal bundle of Y in X is q-concave. From Theorem 5.4 in [4] the proof of Proposizion 4 follows in the same way as in the second part of the proof of Theorem 1. $\square$

The condition that Y has a strictly plurisubharmonic function which is bouded from above is satisfied if Y is an open subset of an hyperconvex space and in particular if Y is a relatively compact open subset of a Stein space. But when the space X, or at least Y, is contained in a Stein space stronger results can be obtained by the following easy remark. The restriction of a 1-positive linear space on X to an open subset U of X is 1-positive. Therefore the trivial line bundle on any open subset of a Stein space is 1-positive. It follows easily that any vector bundle generated by global sections is 1-positive on an open subset of a Stein space of bounded dimension. Therefore we obtain immediately from Proposition 2 and the standard technique used in the proof of Theorem 2 and Proposition 4 the following theorem.

THEOREM 2. *Let* X *be a r-complete open submanifold of a Stein space and* Y *be a closed analytic subspace of* X. *Suppose that* Y *is a locally complete intersection in* X *of codimension* q *with normal bundle generated by global sections. Then* $X \setminus Y$ *is a* $(q + r - 1)$*-complete manifold.*

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