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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Systems of convolution equations and LAU–spaces.* Nota di DANIELE C. STRUPPA (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Dato un sistema omogeneo di equazioni di convoluzione in spazi dotati di strutture analiticamente uniformi, si forniscono condizioni per ottenere teoremi di rappresentazione per le sue soluzioni.

Let $E = E(\mathbf{R}^n)$ be the space of infinitely differentiable functions, topologized with the usual uniform convergence topology; let $m \in E'(\mathbf{R}^n)$ be a distribution with compact support. In [1] it is shown that if m is slowly decreasing (we refer the reader to [1] for the definition), then every $f \in E(\mathbf{R}^n)$ satisfying the convolution equation $m * \check{f}(x) = 0$ $(\check{f}(x) = f(-x))$ can be represented as

$$f(x) = \sum_{k=1}^{+\infty} \left(\sum_{j=\mathrm{J}_k} \int_{\mathrm{V}_j} \partial_j (e^{-ix\cdot z})\, \mathrm{d}v_j(z) \right),$$

where V_j $(j \in \mathrm{J})$ are closed sets contained in $\mathrm{V} = \{z \in \mathbf{C}^n : \hat{m}(z) = 0\}$, $\{\mathrm{J}_k\}_{k=1}^{+\infty}$ is a partition of the index set J into finite subsets, ∂_j are partial differential operators with constant coefficients and $\mathrm{d}v_j$ are complex valued Radon measures supported in V_j. In [1] this result is also obtained, with small modifications, for the case of systems with one unknown function.

In this note we announce some extensions of those previous results; the proofs will appear elsewhere.

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We begin with some notations: given a functional space X, we will denote by $\hat{X}'$ the space of the Fourier transforms of the elements of the dual X' of X; moreover if $p(z)$ is a suitably nice plurisubharmonic function on $\mathbf{C}^n$ (see [5] for the conditions which $p(z)$ must satisfy), we will call $A_p(\mathbf{C}^n)$ the space of entire functions $F(z)$ such that, for some $A, B > 0$ (constants depending on F), it is

$$|F(z)| \leq A \exp(Bp(z)).$$

Example 1: If $X = E(\mathbf{R}^n)$, then $\hat{X}'$ is isomorphic to $A_p(\mathbf{C}^n)$ for $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$; similarly, if $X = A(\mathbf{C}^n)$, the space of entire functions, the isomorphism holds with $p(z) = |z|$.

The results of [1] can be extended to obtain:

THEOREM 1. *Let* X *be a space such that* $\hat{X}' = A_p(\mathbf{C}^n)$ *for some plurisubharmonic function* $p(z)$. *Consider the* $N \times r$ $(1 \leq r \leq N)$ *homogeneous system of convolution equations*

$$m_{11} * \check{f}_1 + \cdots + m_{1r} * \check{f}_r = 0$$
$$m_{N1} * \check{f}_1 + \cdots + m_{Nr} * \check{f}_r = 0,$$

for $f_i \in X$, $m_{ik} \in X'$.

Suppose $F = [m_{ik}]^t$ *satisfies slowly decreasing conditions similar to those introduced in* [1]. *Then we can find closed sets* V_{ij} $(i = 1, \cdots, r; j \in J^{(i)})$, *partitions of the index sets* $J^{(i)}$ *into finite subsets* $J_k^{(i)}$ *and a map* ρ *defined on* $(A_p)^r$, *whose definition is analogous to the one given by Ehrenpreis in* [2], *so that each solution of the previous system can be represented as*

$$f_i(x) = \sum_{k=1}^{+\infty} \left(\sum_{j \in J_k^{(i)}} \int_{V_{ij}} [\rho\{e^{-ix \cdot z}\}]_i \, dv_{ij}(z) \right),$$

for dv_{ij} *complex valued Radon measures supported in* V_{ij}, *and where the series and the integrals converge in* X.

The proof of theorem 1 follows the one given by Berenstein and Taylor in [1], for the case of only one unknown function, with the modifications introduced by the fact that we are now dealing with matrices instead of vectors. Let us denote the system which f satisfies by $T\check{f} = 0$. We are interested in studying the space $M = \{f \in X^r : T\check{f} = 0\}$: denote by I the module generated by the matrix F in $[A_p(\mathbf{C}^n)]^r$, where p is such that $\hat{X}' = A_p(\mathbf{C}^n)$. Using the fact that F is a slowly decreasing matrix, one can prove (the procedure, after [1], is standard, although in this case one has to deal with several technical problems connected with the fact that it is necessary to work with modules instead of ideals as in [1]) that I is closed. Then, by standard functional analysis, one shows that M is the dual space to $(\hat{X}')^r/I = [A_p(\mathbf{C}^n)]^r/I$. At this point, in order to conclude the proof, it is necessary to describe the space $[A_p(\mathbf{C}^n)]^r/I$: this is done by identifying this space with a space of functions analytic on the " vector multiplicity variety " associated to the module I (see

[2] for the necessary definitions); this identification is in fact obtained via the local restriction map ρ which appears in the statement of the theorem. Finally, given $f \in \mathrm{M}$, we can find a corresponding element $\mathbf{F} \in (\mathrm{A}_p)^r/\mathrm{I}$, and the theorem is proved by applying the Riesz representation theorem and observing that it is

$$f(x) = \mathbf{F}\left(\rho\left\{e^{-ix\cdot z}\right\}\right),$$

where $\{e^{-ix\cdot z}\}$ denotes the r-vector whose components are all equal to $e^{-ix\cdot z}$.

As an example we notice that if we consider a two by two system of convolution equations as in theorem one, then it is possible to show that

$$\mathrm{V}_{1j} \subset \mathrm{V}_1 = \{z \in \mathbf{C}^n : \hat{m}_{11}(z) = \hat{m}_{21}(z) = 0\}$$

and

$$\mathrm{V}_{2j} \subset \mathrm{V}_2 = \{z \in \mathbf{C}^n : \hat{m}_{11}(z)\,\hat{m}_{22}(x) - \hat{m}_{12}(z)\,\hat{m}_{21}(z) = 0\},$$

so that we will require, in particular, that both V_1 and V_2 must be slowly decreasing varieties in the sense of [1]. Before presenting other results, I would like to briefly describe how one can show that I is closed in $(\mathrm{A}_p)^r$. What one tries to prove, is that if f belongs locally to the module generated by F (i.e. for each $z \in \mathbf{C}^n$, there exists $\mathrm{U}(z)$ in which $f = \mathrm{F}\cdot g$ for $g \in (\mathrm{A}(\mathrm{U}))^r$), then $f \in \mathrm{I}$. In order to achieve this result, we start by constructing the varieties associated to F, and we cover them with a family C of open sets W which must satisfy two conditions:

i) it must be possible to apply the Jacobi interpolation formula (see [1]) on each of them (this actually translates into very technical conditions for the W);

ii) $\bigcup_{\mathrm{W} \in \mathrm{C}} \mathrm{W}$ should be such to enable us to apply the semi-local to global interpolation theorem (see [1]).

Of course such a covering does not always exist, and one of the main roles of the slowly decreasing condition is to insure exactly that. At this point one first constructs h locally in each W, so that $f = \mathrm{F}\cdot h$ in W, using a slight generalization of the Jacobi interpolation formula, then corrects the resulting cochain, via the so called generalized Koszul complex (see its description in [6], where it is used for purposes very closed to ours), and finally extends the resulting vector from $\bigcup_{\mathrm{W} \in \mathrm{C}} \mathrm{W}$ to all of $\mathbf{C}^n$, by employing the techniques introduced by Hormander in [4].

As a consequence of the methods developed to prove the previous results, it is also possible to obtain a sort of Fredholm alternative theorem for systems of convolution equations:

THEOREM 2. *Let* $\mathrm{T} : \mathrm{X}^r \to \mathrm{X}^\mathrm{N}$ *be a map given by convolution product with a slowly decreasing matrix* $\mathrm{T} = (\mathrm{T}_{ij})$ *of elements of* X', *for* X *as before. Then, for* $v \in \mathrm{X}^\mathrm{N}$, *the equation*

$$\mathrm{T}u = v$$

has a solution $\boldsymbol{u} \in \mathrm{X}^r$ *if and only if* $\boldsymbol{v}$ *satisfies the so called compatibility conditions, i.e.*

$$\mathbf{Q}\boldsymbol{v} = 0$$

whenever $\mathbf{Q}$ *belongs to the module of relations of the rows of* T *in* X′.

We notice that results of this same kind have already been proven, for T a partial differential operator, in [2] and in [4].

We wish to conclude pointing out that we have been able to generalize the previous results to a large class of spaces. In particular, by using a definition of LAU-spaces due to Hansen, [3], which is equivalent to the original definition of Ehrenpreis, [2], but more convenient for our purposes, we succeeded in proving a representation theorem (for systems of convolution equations), for the most important LAU-spaces, among which we recall $E(\mathbf{R}^n)$, $\mathrm{A}(\mathbf{C}^n)$ and $\mathscr{D}'(\mathbf{R}^n)$; we observe, incidentally, that the methods of [3] do not succeed in studying the case of $\mathscr{D}'$. The basic observation which enabled us to extend the representation theorem to such spaces is the fact that every LAU-space can be, roughly speaking, identified with a projective limit or with an inductive limit of spaces of the form A_p. This fact was probably well known, but the new definition of Hansen makes it particularly easy to exploit. Of course this result is of particular interest, because it is exactly for LAU-spaces that Enhrenpreis proved his Fundamental Principle (see [2]), i.e. the representation theorem for systems of partial differential equations with constant coefficients, of which our theorem 1 can be considered a natural extension.

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