Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Massimo Tessarotto

A model collision operator for the drift Fokker-Planck equation for applications to transport problems in magnetoplasmas

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **71** (1981), n.3-4, p. 29–35.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1981_8_71_3-4_29_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1981.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Ferie 1981 (Settembre-Ottobre)

(Ogni Nota porta a pie' di pagina la data di arrivo o di presentazione)

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Fisica matematica. — *A model collision operator for the drift Fokker–Planck equation for applications to transport problems in magnetoplasmas.* Nota (*) di MASSIMO TESSAROTTO (**), presentata dal Socio D. GRAFFI.

RIASSUNTO. — Nella Nota viene presentato un modello di operatore di collisione per l'equazione cinetica di (deriva) di Fokker–Planck, valido per magnetoplasmi quiescenti immersi in configurazioni idromagnetiche di equilibrio simmetriche.

Principale caratteristica del presente modello è di consentire – contrariamente ad operatori di collisione approssimati in precedenza proposti da altri autori – la determinazione di variabili macroscopiche rilevanti includendo correzioni del primo ordine in termini di uno sviluppo perturbativo in funzione di un opportuno parametro adimensionale (Δ) che caratterizza le inomogeneità della configurazione magnetica.

1. INTRODUCTION

In kinetic theory, the adoption of approximate models for the (Boltzmann) collision operator results, in many cases, convenient due to the relevant simplification which they afford. However, it is obvious that their practical usefulness for the computation of relevant macroscopic dynamical variables for a given mechanical system, depends from their accuracy and range of validity, which—unfortunately—results often hard to assess 'a priori'.

(*) Pervenuta all'Accademia il 7 ottobre 1981.

(**) FOM–Instituut voor Plasmafysica, Rijnhuizen, Nieuwegein, The Netherlands; permanent address: Istituto di Meccanica, Università degli Studi, Trieste, Italy.

In the past various models have been proposed on the basis of simple physical requests (in particular conservation laws to be fulfilled by the operator) as well as mathematical simplicity. A well known example is the model pointed out by Bhatnagar, Gross and Krook in a celebrated paper [1], which for its inherent simplicity (it is a linear operator, which in its most elaborate forms may contain momentum and energy restoring coefficients, and conserves the total mass as well) is frequently adopted in the most disparate problems, ranging from rarified gas dynamics (see, e.g., Ref. [2]) to plasma dynamics. A further example to be recalled is the so-called Lorentz model (see for example Ref. [3]), which results formally from a mutilation of the Fokker-Planck collision operator (to which the Boltzmann operator can be reduced under the assumption of 'soft' collision, as appropriate to plasmas) and the addition of a momentum restoring term. More recent models adopted to describe transport processes in magnetoplasmas are more elaborate versions of the Lorentz approximation, which include also an approximate description of self-collisions and contain momentum and energy restoring coefficients (in analogy with the BGK model) [4–7].

All such model collision operators contain limitations. For example, it is well known that the BGK operator is not appropriate to describe weakly collisional magnetoplasmas which, in the absence of turbulent perturbations and for weakly inhomogeneous magnetic configurations, are essentially dominated (e.g. for closed configurations) by the dynamics of trapped particles. Thus in this case particle and kinetic energy transport are essentially produced by effect 'localized' in velocity space (in the domain of trapped articles [4]).

Analogously the adoption of this model seems questionable for the investigation of stability and/or transport problems for (weakly) collisional magnetoplasmas in the presence of weakly turbulent perturbations. Similar objections may be cited for the Lorentz model (it neglects self-collisions and its validity is further in question for multi-species systems) as well for the more advanced models of Ref.s [4–]: they yield predictions accurate at most to leading-order in terms of a perturbative expansion with respect to an adimensional parameter Δ which specifies the "degree of localization". A possible definition for Δ is:

$$(1) \qquad \Delta = \delta \mid 1 - \delta \mid^2 \qquad \text{with} \quad \delta = \langle (1 - B/B_{\max})^{\frac{1}{2}} \rangle$$

where $\Delta \ll 1$ by assumption, e.g. either $0 < \delta \ll 1$ or $\mid 1 - \delta \mid \ll 1$ (B_{max} is the absolute maximum of B on a given isobaric surface and the brackets denote an appropriate averaging operation on the same isobaric surface), thus since the 'width' of the domain of trapped particles is proportional to δ we see that if $\delta \ll 1$ then $\Delta \approx \delta$ and the localization occurs in the domain of trapped particles, whereas if $\delta \to 1$, $\Delta \approx \mid 1 - \delta \mid^2$ the localization results in the complementary domain (of circulating particles). It should be remarked, however, that whereas the model operator of Ref. [7] yields the correct results in both limits (1), those

(1) At least for the computation of the partilec fluxes.

of Ref.s [5] and [6] are satisfactory only for configurations with weakly inhomogenous magnetic field ($\delta \ll 1$). Similar is the situation of the model presented in Ref. [4] (which refers to the special case of a two-component plasma), e.g. it yields predictions accurate at most to leading order with respect to Δ and for $\delta \ll 1$ (not for $\delta \to 1$).

Apart from such essential drawbacks, which limit the accuracy of such models or their range of validity, it seems that in some instances their introduction lacks mathematical (as well as physical) justification. In particular a reliable model should be constructed on the basis of rigorous criteria and its predictions should be checked once for all.

The essential mathematical criterion which shall be adopted in the sequel, to construct an esplicit model collision operator, is simply that it should yield, with prescribed degree of accuracy, appropriate moments of the distribution function, to be chosen in accordance with the specific class of problems.

More precisely, for quiescent magnetoplasmas, possible relevant moments are the particle and kinetic energy fluxes, which appear in the continuity and energy balance equations. However other choices may be relevant as well. For example, in the case of turbulent magnetoplasmas, further quantities of interest are the growth rate and the wave number of the fastest growing perturbation.

A possible mathematically meaningful request would therefore be that a given model collision operator delivers the correct values of such moments up to order, say, $O(\Delta^p)$ with $p = 1, 2, \cdots, k$, after expanding each relevant physical quantity in power series of Δ.

In the sequel we intend to illustrate a model collision operator of this class which shall be discussed elsewhere in greater detail [8]. More precisely it is constructed in such a way to yield, up to order $O(\Delta)$, the correct particle and kinetic energy fluxes for a quiescent (weakly) collisional magnetoplasma. Furthermore, for definitess, we shall limit ourselves to a class of hydromagnetic equilibria which are symmetric, e.g. exibit at least one ignorable spatial coordinate.

It is felt that, although obtained for a specific problem, the actual relevance of the model lies not only in its detailed applications but has mathematically interesting features and results susceptible of further developments and generalizations.

2. A MODEL COLLISION OPERATOR FOR THE DRIFT FOKKER-PLANCK EQUATION

Let us write down, for definitess, the drift Fokker-Planck equation in the form [3]:

$$v_{\parallel} \boldsymbol{n} \cdot \nabla g_s = C_s (f_0 \mid \bar{f}_1) \qquad \text{with} \quad \bar{f}_{1,s} = g_s + g_s^{(D)} \tag{2}$$

and

$$g_s^{(D)} = \varkappa_s \, v_{\parallel} f_{0,s} \left(A_{1s} + x_s^2 A_{2s} \right) . \tag{3}$$

Here the notations are standard (see, e.g., Ref.s [8, 9]) and, in particular, $\tilde{f}_{1,s}$ is the perturbation of the local maxwellian distribution $f_{0,s}$ and $C_s(f_0 \mid \tilde{f}_1)$ the linearized Fokker-Planck collision operator in the Landau form, recalled in the Appendix. We shall confine our investigation of Eq. (2) to the limit $\nu_{s,\mathrm{eff}}/\omega_{b,s} \ll 1$ (so-called "weakly collisional regime" where $\nu_{s,\mathrm{eff}}$ and $\omega_{b,s}$ are respectively an appropriate effective collision frequency and the bounce frequency characterizing the particle unperturbed motion).

In order to construct a model collision operator for $C_s(f_0 \mid \tilde{f}_1)$ (denoted $C_{M,s}(f_0 \mid \tilde{f}_1)$, in accordance with the program previously outlined, we shall require: *a*) that it yields, up to order $O(\Delta)$ at least, the correct particle and kinetic energy fluxes; *b*) that it is consistent with fundamental physical principles: hence it must conserve momentum and total kinetic energy, as well as the mass; in addition it must yield strictly ambipolar particles fluxes (as follows, for symmetric hydromagnetic equilibria, from the request of momentum conservation), namely:

$$\sum_s e_s \Gamma_{1s} = 0 \tag{4}$$

where e_s is the electric charge and Γ_{1s} the particle flux across and isobaric surface.

We introduce, thus, a parameter-dependent family of linear operators, $C_{M,s}(f_0 \mid \tilde{f}_1)$, denoting by $C_{M,s}(f_0 \mid \tilde{f}_1)$ its contribution of order $O(\Delta^p)$ $(p = 0, 1)$. In particular, for circulating particles $(0 \leqq \lambda < 1/B_{\max}$, with λ the usual pitch-angle variable) we define:

$$C_{M,s}^{(0)}(f_0 \mid \tilde{f}_1) = K_0(\tilde{f}_{1,s}^{(0)}(C^{(0)})) \tag{5}$$

$$\begin{aligned} C_{M,s}^{(1)}(f_0 \mid \tilde{f}_1) = {} & K_0(\tilde{f}_{1,s}^{(1)}(C^{(1)})) + K_1(\tilde{f}_{1,s}^{(0)}(C^{(0)})) + \\ & + v_\parallel f_{0,s}(1 - m_s v^2/T_s)\frac{1}{M_{1s}}\int d^3 v v_\parallel \sum_k Z_{1ks}\tilde{f}_{1,k}^{(0)}(C^{(0)}) + \\ & + v_\parallel f_{0,s}(1 - v^2/v_s^2)\frac{1}{M_{Es}}\int d^3 v v_\parallel \sum_k Z_{Eks}\tilde{f}_{1,k}^{(0)}(C^{(0)}) \end{aligned} \tag{6}$$

whereas, for trapped particles $(1/B_{\max} \leqq \lambda \leqq 1/B)$:

$$\begin{aligned} C_{M,s}(f_0 \mid \tilde{f}_1) = {} & K_s(g_s^{(D)}) + K_1(g_s^{(D)}) + \\ & + v_\parallel f_{0,s}(1 - m_s v^2/T_s)\frac{1}{M_{1s}}\int d^3 v v_\parallel \sum_k Z_{1ks}\tilde{f}_{1,s}^{(0)}(C^{(0)}) + \\ & + v_\parallel f_{0,s}(1 - v^2/v_s^2)\frac{1}{M_{Es}}\int d^3 v v_\parallel \sum_k Z_{Eks}\tilde{f}_{1,k}^{(0)}(C^{(0)}) . \end{aligned} \tag{7}$$

Here we have introduced the positions:

$$\begin{aligned} & T_s = 7\, T_{0,s} \quad ; \quad v_s = (5/2)^{\frac{1}{2}} v_{th,s} \\ & M_{1s} = \frac{1}{7} N_{0,s} v_{th,s}^2 \quad ; \quad M_{Es} = \frac{1}{2} N_{0,s} T_{0,s} v_{th,s}^2 \end{aligned} \tag{8}$$

to assure momentum and energy conservation. Furthermore the operators K_0, K_1 and the energy functions Z_{1ks} and Z_{Eks} are defined in the Appendix. Finally the real constants $C^{(i)}$ $(i = 0, 1)$, are introduced by defining:

$$\bar{f}^{(i)}_{1,s}(C^{(i)}) = \bar{f}^{(i)}_{1,s} + m_s v_{\parallel} C^{(i)} f_{0,s} \tag{9}$$

and are uniquely determined by Eq. (4).

We note, in particular, that since in the case of trapped particles $\bar{f}_{1,s} = g_s^{(D)} + O((\nu_{s,\mathrm{eff}}/\omega_{b,s})^2)$ and the 'width' of their domain is proportional to δ, we expect that their contribution to particle and kinetic energy fluxes results of order $O(\delta^3)$ for $\delta \to 0$ and $O(|1-\delta|^0)$ for $\delta \to 1$.

Let us examine briefly the main features of the present model collision operator. The term $C^{(0)}_{M,s}(f_0 \mid \bar{f}_1)$ (Eq. (5)) contains in particular (for $C^{(0)} = 0$) the usual pitch-angle-scattering part of the Fokker-Planck collision operator, analogously to the Lorentz model, with the addition of a momentum restoring term. Thus apart terms of order $O(\delta)$ (if $\delta \to 0$) it is equivalent to the models proposed in Ref.s [4–7]. The remaining contributions (Eq.s (6) and (7)) have no previous counterpart. However, in Ref. [7] a model operator has been adopted to take into account the contribution of trapped particles in the limit $\delta \to 1$. Their operator yields for trapped particles—in contrast to ours, see, e.g., Eq. (7)—corrections of order $O(\delta^2)$ (instead of $O(\delta^3)$) to the particle and kinetic energy fluxes and thus appears inconsistent with the previous conclusions. Thus their model seems inadeguate for the purpose of determining contributions of order higher than $O(\delta)$.

We point out that our model collision operator yields indeed the correct expressions for the particle and kinetic energy fluxes up to order $O(\Delta)$. The detailed proof of this statement shall be reported elsewhere [8].

As final comment, we stress that the present model is also consistent with the request of local thermodynamic equilibrium, invoked to derive Eq. (2), namely $f_{0,s}$ is a solution of the homogeneous equation:

$$C_{M,s}(f_0 \mid f_0) = 0 \tag{8}$$

as should be expected.

Acknowledgements. — This work was performed in the framework of the author's participation in the activities of the 'Gruppo Nazionale per la Fisica Matematica del C.N.R.' (Italy) as well as under the Euratom-FOM Association agreement with financial support from ZWO and Euratom.

The author is indebted to Prof. Engelmann for helpful comments and hospitality at the FOM-Istituut voor Plasmafysica.

APPENDIX

Here we recall briefly the usual definition for the Fokker–Planck collision operator in the Landau form to express it in the $\boldsymbol{v}$–space coordinate v, λ and θ ($\lambda = \mu/\mathrm{E}$ with $\mu = v_{\perp}^2/2\mathrm{B}$, $\mathrm{E} = v^2/2$ and θ is an azimuth in a plane $\perp$ to $\mathbf{B}$). From. Eq. (25) results:

$$\sum_k \mathrm{C}_{sk}(f_{0,k} \mid \bar{f}_{1,s}) = \mathrm{K}_0(\bar{f}_{1,s}) + \mathrm{K}_1(\bar{f}_{1,s}) \tag{A.1}$$

$$\mathrm{C}_{sk}(\bar{f}_{1,k} \mid f_{0,s}) = q_{sk} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}^3 v' \frac{\partial^2 u}{\partial \boldsymbol{v}\, \partial \boldsymbol{v}} \cdot \left\{ \bar{f}_{1,k} \frac{\partial}{\partial \boldsymbol{v}} f_{0,s} - \frac{m_s}{m_k} \frac{\partial}{\partial \boldsymbol{v}'} \bar{f}_{1,k} f_{0,s} \right\} \tag{A.2}$$

$$\mathrm{K}_0(\bar{f}_{1,s}) = \frac{\mathrm{U}_s(v)}{\mathrm{B}v^3} (1 - \lambda \mathrm{B})^{\frac{1}{2}} \frac{\partial}{\partial \lambda} \left\{ \lambda (1 - \lambda \mathrm{B})^{\frac{1}{2}} \frac{\partial}{\partial \lambda} \bar{f}_{1,s} \right\} \tag{A.3}$$

$$\mathrm{K}_1(\bar{f}_{1,s}) = \left\{ \frac{1}{2v^2} \frac{\partial}{\partial v} - \lambda (1 - \lambda \mathrm{B}) \frac{\partial}{\partial \lambda} \right\} \left\{ v \frac{\partial}{\partial v} \mathrm{U}_s(v) \exp\{\alpha_s(v)\} \cdot \right. \tag{A.4}$$

$$\left. \cdot \left\{ v \frac{\partial}{\partial v} - \lambda (1 - \lambda \mathrm{B}) \frac{\partial}{\partial \lambda} \right\} (\bar{f}_{1,s} \exp\{-\alpha_s(v)\}) \right\}$$

$$\mathrm{U}_s(v) = \frac{2}{\pi^{\frac{1}{2}}} \sum_k q_{sk} \mathrm{N}_{0,k} \left(x_k^{-1} \exp\{-x_k^2\} + 2 \left(1 - \frac{1}{2x_k^2}\right) \int_0^{x_k} \mathrm{d}t \exp - t^2 \right) \tag{A.5}$$

$$\alpha_s(v) = -2 \int_0^v \mathrm{d}v'\, v' \frac{\sum_k q_{sk} v_{th,k}^2 \left(\int_0^{v'} \mathrm{d}v''\, f_{0,k} - v' f_{0,k} \right) m_s/m_k}{\sum_k q_{sk} v_{th,k}^4 \left(\int_0^{v'} \mathrm{d}v''\, f_{0,k} - v' f_{0,k} \right)} . \tag{A.6}$$

Finally in Eq.s (6) and (7) $\mathrm{Z}_{\mathrm{F}sk}$ for $\mathrm{F}_s = 1$, E_s ($\mathrm{E}_s = m_s v^2/2$) are the integrals:

$$\mathrm{Z}_{1ks} = x_s^{-2} \left(f_{0,s} - \frac{1}{v} \int_0^v \mathrm{d}v' f_{0,k} \right) \Xi_{sk} \tag{A.7}$$

$$\mathrm{Z}_{\mathrm{E}ks} = -4\pi q_{sk} \mathrm{T}_{0,s} \left(-3(1 + m_s/m_k) \left\{ f_{0,s} \left(1 + \frac{11}{6} x_s^{-2}\right) - \frac{11}{6} x_s^{-2} \frac{1}{v} \int_0^v \mathrm{d}v' f_{0,s} \right\} + \right. \tag{A.8}$$

$$\left. + 3 \frac{m_s}{m_k} x_s^{-2} \left\{ f_{0,s} - \frac{1}{v} \int_0^v \mathrm{d}v' f_{0,s} \right\} \right) \cdot$$

$$\alpha_{sk} = v_{th,k}^2 / v_{th,s}^2 \tag{A.9}$$

$$\Xi_{sk} = 4\pi (1 + m_s/m_k) q_{sk} \tag{A.10}$$

with $q_{sk} = 2\pi e_s^2 e_k^2 \ln \Lambda / m_s^2$.

REFERENCES

[1] P. L. BHATNAGAR, E. P. GROSS and M. KROOK (1954) - « Phys. Rev. », *94*, 511.
[2] C. CERCIGNANI (1975) - *Theory and Application of the Boltzmann Equation*, « Scottish Academic Press », Edinburgh and Elsevier, New York.
[3] P. H. RUTHERFORD (1970) - « Phys. Fluids », *13*, 482.
[4] M. N. ROSENBLUTH, R. D. HAZELTINE and F. L. HINTON (1972) - « Phys. Fluids », *15*, 116.
[5] L. M. KOVRIZHNIKH (1970) - Trieste Report IC/70/86, p. 25.
[6] J. W. CONNOR (1973) - « Plasma Phys. », *15*, 765.
[7] S. P. HIRSHMAN, D. J. SIGMAR and J. F. CLARKE (1976) - « Phys. Fluids », *19*, 656.
[8] M.TESSAROTTO (1981) - FOM-Instituut voor Plasmafysica (Nieuwegein, The Netherlands), Int. Rep. 81/033, submitted to « Meccanica »,
[9] M. TESSAROTTO. (1981) - FOM-Instituut voor Plasmafysica (Nieuwegein The Netherlands), Int. Rep. 81/024, submitted to « Meccanica »,