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Meccanica razionale. — *Material constraints in continuum mechanics.* Nota di STUART S. ANTMAN (*), presentata (**) dal Socio straniero C. TRUESDELL.

RIASSUNTO. — Si dimostra che ci sono valide ragioni per considerare la teoria standard dei vincoli interni, nella meccanica dei continui, insufficientemente generale. In particolare, con l'unica eccezione dell'iperelasticità, l'extra-stress dovrebbe dipendere anche dai moltiplicatori di Lagrange, cioè, dallo stress che non effettua lavoro (virtuale).

In this note we show that there are physically sound reasons suggesting that the accepted doctrines for the treatment of material (or internal) constraints in continuum mechanics are insufficiently general. We then explain how the requisite level of generality can be attained and indicate some of the important consequences arising from this generality.

Let $C(x, t)$ be the Cauchy-Green deformation tensor at material point x of a body at time t. We define $C^t(x, \tau) = C(x, t-\tau)$ for $\tau \geq 0$. Let $S(x, t)$ be the second Piola-Kirchhoff stress tensor at (x, t). Then the frame indifferent mechanical constitutive equation for an unconstrained simple material point x is

$$S(x, t) = \hat{S}(C^t(x, \cdot), x) \tag{1}$$

where $\hat{S}$ is a prescribed symmetric tensor-valued functional (cf. [8, Sec. 26]).

We now study the corresponding constitutive relations when $C(x, \cdot)$ is no longer free to range over the space V of positive-definite symmetric tensors but must satisfy a set of independent, frame-indifferent constraints of the form

$$\varkappa_k(C, x) = 0 \quad , \quad k = 1, \cdots, K \leq 6. \tag{2}$$

Here the $\varkappa_k$ are prescribed function with $\varkappa_k(\cdot, x)$ continuously differentiable on V. For $K = 6$, the deformation at x is rigid, so the only interesting cases are those for which $K < 6$. Under these conditions (2) defines a $6 - K$ dimensional manifold $M = M(x)$ in V. In a sufficiently small neighborhood N in V of any point C_0 of M we can introduce a system of curvilinear coordinates $(u, v) = u^1, \cdots, u^{6-K}, v^1, \cdots, v^K)$ with u serving as coordinates for M, with v orthogonal to u, and with $v = 0$ for points C on $M \cap N$. Thus any C in N is identified with its coordinate (u, v) by means of the function $N \ni C \mapsto (\bar{u}(C), \bar{v}(C))$, whose inverse is denoted $(u, v) \mapsto \bar{C}(u, v)$.

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A basis for N is $(\bar{C}_u, \bar{C}_v) = (\partial\bar{C}/\partial u^1, \cdots, \partial\bar{C}/\partial u^{6-K}, \partial\bar{C}/\partial v^1, \cdots, \partial\bar{C}/\partial v^K)$. We introduce a basis in span $\{\partial\bar{C}/\partial u^1, \cdots, \partial\bar{C}/\partial u^{6-K}\}$ dual to $\{\partial\bar{C}/\partial u^1, \cdots, \partial\bar{C}/u^{6-K}\}$ and let the components of S with respect this dual basis be denoted $m = (m_1, \cdots, m_{6-K})$. We introduce a basis in span $\{\partial\bar{C}/\partial v^1, \cdots, \partial\bar{C}/v^K\}$ dual to $\{\partial\bar{C}/\partial v^1, \cdots, \partial\bar{C}/\partial v^K\}$ and let the components of S with respect to this dual basis be denoted $n = (n_1, \cdots, n_k)$.

The Constraint Principle of Truesdell and Noll [9, Sec. 30], generalizing the work of Ericksen and Rivlin [6] (and subsequently generalized to thermo-mechanical problems by Green, Naghdi and Trapp [7]), asserts that the mechanical constitutive equations for a simple material point x satisfying (2) consist of (2) and

$$\text{(3)} \qquad S(x, t) = -\sum_1^K l_k(x, t) \frac{\partial \varkappa_k}{\partial C}(C(x, t), x) \sqrt{\det C(x, t)} + \hat{S}(C^t(x, \cdot), x)$$

where $l = (l_1, \cdots, l_K)$ is a function not prescribed by constitutive equations and where $\hat{S}$ need only be defined for histories satisfying (2). In terms of the variables u, v, m, n, the constraints (2) reduce to

$$\text{(4)} \qquad v = 0$$

and (3) reduces to an equation of the form

$$\text{(5)} \qquad m(x, t) = \hat{m}(u^t(x, \cdot), x).$$

By choosing the coordinates v appropriately we could identify l with n of (3). n is not prescribed by a constitutive equation. These formulations are compatible with the principle that the stresses are only defined to within a stress that does no (virtual) work in any deformation satisfying the constraints. The particular form of (3) generalizes that coming from the variational characterization of the equilibrium configuration of a hyperelastic body under conservative loads as one that extremizes the potential energy function.

Our main thesis in this note is that (3) or (5) is unduly restrictive: (3) should be generalized by allowing $\hat{S}$ to depend upon l^t as well. In particular, (5) should be replaced by

$$\text{(6)} \qquad m(x, t) = \hat{m}(u^t(x, \cdot), n^t(x, \cdot), x).$$

There is no way to *prove* this thesis. We can merely show that it is physically and mathematically natural and that it is consistent with accepted methods of describing material behavior.

To support this assertion we write the constitutive equation (1) for an unconstrained simple material point in the equivalent (local) form

$$\text{(7)} \qquad m(x, t) = \hat{m}(u^t(x, \cdot), v^t(x, \cdot), x),$$

$$\text{(8)} \qquad n(x, t) = \hat{n}(u^t(x, \cdot), v^t(x, \cdot), x).$$

Let us suppose that $\hat{n}(u^t, \cdot, x)$ is invertible in the sense that (8) is equivalent to an equation of the form

$$v(x, t) = v^{\#}(u^t(x, \cdot), n^t(x, \cdot), x). \tag{9}$$

The substitution of (9) into (7) reduces it to an equation of the form

$$m(x, t) = m^{\#}(u^t(x, \cdot), n^t(x, \cdot), x). \tag{10}$$

Since a real material satisfies constraints (2) only approximately (e.g., water and rubber are only approximately incompressible) we may regard the constrained material as a limiting case of a family of unconstrained materials. In particular, the constitutive equations for a material satisfying the constraints (4) (which are equivalent to (2)) are obtained from (9) and (10) by letting $v^{\#} \to 0$. In this limit n is not prescribed by a constitutive function, while (10) remains unchanged. Moreover (10) is equivalent to (6). (Note that $m^{\#}$ of (10) is a composite function with $\hat{m}$ acting on $v^{\#}$. In any process by which $v^{\#} \to 0$, there is no reason to hold $\hat{m}$ fixed. Thus it is not generally true that the limiting form of $m^{\#}$ is independent of n^t).

Before discussing the supposition that $\hat{n}(u^t, \cdot, x)$ is invertible, let us consider two examples. For an elastic material, (7) and (8) reduce to

$$m(x, t) = \hat{m}(u(x, t), v(x, t), x), \tag{11}$$

$$n(x, t) = \hat{n}(u(x, t), v(x, t), x). \tag{12}$$

The invertibility of $\hat{n}(u, \cdot, x)$ leads to the system

$$v(x, t) = v^{\#}(u(x, t), n(x, t), x), \tag{13}$$

$$m(x, t) = m^{\#}(u(x, t), n(x, t), x) \equiv \hat{m}(u(x, t), v^{\#}(u(x, t), n(x, t), x), x), \tag{14}$$

which is equivalent to (11) and (12) and which is the appropriate specialization of (9) and (10). The constitutive equations for the constrained material, obtained by letting $v^{\#} \to 0$, consist of (4) and (14). For a viscoelastic material of differential type with complexity 1, (7) and (8) reduce to

$$m(x, t) = \hat{m}(u(x, t), u_t(x, t), v(x, t), v_t(x, t), x), \tag{15}$$

$$n(x, t) = \hat{n}(u(x, t), u_t(x, t), v(x, t), v_t(x, t), x). \tag{16}$$

If $\hat{n}(u, u_t, v, \cdot, x)$ is invertible, then we can replace (15) and (16) with the equivalent system

$$v_t(x, t) = \dot{v}^{\#}(u(x, t), u_t(x, t), v(x, t), n(x, t), x), \tag{17}$$

$$m(x, t) = m^{\#}(u(x, t), u_t(x, t), v(x, t), n(x, t), x). \tag{18}$$

Regarding (17) as an ordinary differential equation for $v(x, \cdot)$ we could impose an initial condition and solve it (under favorable conditions) to get a representation of the form (9) for v. This procedure is unnecessary for our present goals. We can simply obtain the equations for the constrained material by letting $\dot{v}^{\#}$ approach 0 and by requiring that there be a real number τ for which $v(x, \tau) = 0$. In this way we recover (4) and we obtain

$$m(x, t) = m^{\#}(u(x, t), u_t(x, t), 0, n(x, t), x) \tag{19}$$

as the specialization of (10).

Our supposition that $\hat{n}(u^t, \cdot, x)$ is invertible can be motivated by three different arguments:

i) For certain classes of materials commonly used constitutive inequalities may ensure such invertibility while prohibiting the invertibility of the function $(\hat{m}(\cdot, \cdot, x), \hat{n}(\cdot, \cdot, x))$. E.g., the Strong Ellipticity Condition of elasticity implies the monotonicity of an appropriate stress component along any given line in deformation space. This fact, together with a suitable growth condition, delivers the deformation on the given line as a function of the corresponding stress component and of the other strains and can justify the process leading from (12) to (13) when $K = 1$. The Strong Ellipticity Condition with growth conditions can be used to show that the constitutive equations of elasticity can be uniquely solved for det C in terms of the stresses and the other strains. Thus the constraint of incompressibility can be handled by our methods (cf. [3]). Similarly the invertibility of $\hat{n}(u, u^t, v, \cdot, x)$ appearing in (16) would follow from growth conditions and from an assumption that is weaker than the requirement that the governing equations of motion be parabolic (but stronger than the consequences of the Clausius-Duhem inequality).

ii) Since we propose to obtain (4), which assigns the unique constant function 0 to v in its dependence on (u^t, n^t), we have no reason to prevent $v^{\#}$ of (9) from also depending uniquely on these same arguments. Moreover, if our constrained constitutive equations are to be the limits of those for *all* possible unconstrained materials, then a fortiori they must be the limits of those for which $v^{\#}$ is uniquely defined.

iii) We could just postulate constrained constitutive equation (4) and (10) as our starting point. That $m^{\#}$ should depend on n^t might be regarded as a consequence of the Principle of Equipresence (cf. [8, Sec. 96]). This principle has not been used in the manner advocated here apparently because (10) does not have the traditional form for a simple material.

It is illuminating to study the imposition of constraints on a hyperelastic material. In terms of the variable u, v, m, n with $u, v \in N$, the constitutive functions $\hat{m}$ and $\hat{n}$ of (11) and (12) for an unconstrained hyperelastic material must satisfy

$$\begin{pmatrix} \partial\hat{m}/\partial u & \partial\hat{m}/\partial v \\ \partial\hat{n}/\partial u & \partial\hat{n}/\partial v \end{pmatrix} \text{ is symmetric}, \tag{20}$$

whence it follows that there exists a stored energy function $\hat{W}$ such that

$$\hat{m} = \partial\hat{W}/\partial u \quad , \quad n = \partial\hat{W}/\partial v . \tag{21}$$

If $\hat{n}(u, \cdot, x)$ has inverse $v^{\#}(u, \cdot, x)$, then we can define a new stored energy function $W^{\#}$ by the Legendre transformation

$$W^{\#}(u, n, x) = -n \cdot v^{\#}(u, n, x) + \hat{W}(u, v^{\#}(u, n, x), x) . \tag{22}$$

Then (22) implies that $v^{\#}$ and $m^{\#}$ (introduced in (14)) satisfy

$$m^{\#} = \partial W^{\#}/\partial u \quad , \quad v^{\#} = -\partial W^{\#}/\partial n . \tag{23}$$

Thus for a hyperelastic material, the functions $m^{\#}$ and $v^{\#}$ must satisfy

$$\begin{pmatrix} \partial m^{\#}/\partial u & \partial m^{\#}/\partial n \\ -\partial v^{\#}/\partial u & -\partial v^{\#}/\partial n \end{pmatrix} \text{ is symmetric.} \tag{24}$$

If we now obtain the constitutive equations for the constrained material merely by letting $v^{\#} \to 0$, then we simultaneously destroy the symmetry of the matrix (24) unless we also require that $\partial m^{\#}/\partial n \to 0$. Thus, if the constitutive equations (4) and (14) are to be obtained by our limit process from those for an unconstrained hyperelastic material, then $m^{\#}$ must be independent of n. The same conclusion also follows from the theory of constraints for the variational problem of extremizing the potential energy functional (which is the sum of the stored energy functional and the potential energy of the applied loads). In this case l or n appear as Lagrange multipliers. The point of these observations is that if $m^{\#}$ is to be independent, of n, then this independence must either be based on an ad hoc, explicit postulate, or else must be justified on the basis of other physical principles. (E.g., thermodynamical principles support the assumption of hyperelasticity, which, as we have just shown, prevents $m^{\#}$ from depending on n). Our findings thus show that the work of Ericksen and Rivlin [6], restricted to hyperelastic materials, cannot be faulted for insufficient generality.

To reinforce these ideas and to show that the generality we advocate is not vacuous as it is for hyperelastic materials, we study the imposition of constraints of the form (4) on a thermoviscoelastic material governed by unconstrained constitutive equations of the form

$$v_t(x, t) = \dot{v}^{\#}(u(x, t), u_t(x, t), v(x, t), n(x, t), \theta(x, t), g(x, t), x), \tag{25}$$

$$m(x, t) = m^{\#}(u(x, t), u_t(x, t), v(x, t), n(x, t), \theta(x, t), g(x, t), x), \tag{26}$$

$$q(x, t) = q^{\#}(u(x, t), u_t(x, t), v(x, t), n(x, t), \theta(x, t), g(x, t), x), \tag{27}$$

$$\psi(x, t) = \psi^{\#}(u(x, t), u_t(x, t), v(x, t), n(x, t), \theta(x, t), g(x, t), x), \tag{28}$$

$$\eta(x, t) = \eta^{\#}(u(x, t), u_t(x, t), v(x, t), n(x, t), \theta(x, t), g(x, t), x), \tag{29}$$

where θ is the temperature, g is the temperature gradient, q is the heat flux vector, ψ is the Helmholtz free energy per unit reference volume, and η is the entropy density per unit volume. Note that (25) and (26) generalize (17) and (18). We suppose that the constitutive functions (25)–(29) satisfy the Clausius-Duhem inequality (cf. [4, 9])

$$(30) \qquad \psi_t^{\#} + \eta^{\#}\,\theta_t + \boldsymbol{m}^{\#}\cdot\boldsymbol{u}_t + \boldsymbol{n}\cdot\dot{\boldsymbol{v}}^{\#} + \boldsymbol{q}^{\#}\cdot\boldsymbol{g}/\theta \geq 0 \qquad \forall \boldsymbol{u}, \boldsymbol{u}_t, \boldsymbol{v}, \boldsymbol{n}, \theta, \boldsymbol{g}$$

where the arguments of the functions $\dot{\boldsymbol{v}}^{\#}, \cdots, \eta^{\#}$ are those indicated in (25)–(29). (We can regard (30) as a mere constitutive restriction if we choose not to take it as the correct form of the Second Law of Thermodynamics). We obtain the constitutive equations for a material satisfying (4) by setting $\dot{\boldsymbol{v}}^{\#} = 0$ and deleting the argument $\boldsymbol{v}$ from the remaining constitutive functions. In this case we drop the term $\boldsymbol{n}\cdot\dot{\boldsymbol{v}}^{\#}$ from (30).

By the standard procedure for treating (30) (cf. [4], [9, Sec. 96]) we find for the constrained material that

$$(31) \qquad \partial\psi^{\#}/\partial\boldsymbol{u}_t = 0 \quad , \quad \partial\psi^{\#}/\partial\boldsymbol{n} = 0 \quad , \quad \partial\psi^{\#}/\partial\boldsymbol{g} = 0 \, ,$$

$$(32) \qquad \boldsymbol{m}^{\#}(\boldsymbol{u}, 0, \boldsymbol{n}, \theta, 0, \boldsymbol{x}) = \partial\psi^{\#}(\boldsymbol{u}, \theta, \boldsymbol{x})/\partial\boldsymbol{u} \, ,$$

$$(33) \qquad [\boldsymbol{m}^{\#}(\boldsymbol{u}, \boldsymbol{u}_t, \boldsymbol{n}, \theta, \boldsymbol{g}, \boldsymbol{x}) - \partial\psi^{\#}(\boldsymbol{u}, \theta, \boldsymbol{x})/\partial\boldsymbol{u}]\cdot\boldsymbol{u}_t + \\ + \boldsymbol{q}^{\#}(\boldsymbol{u}, \boldsymbol{u}_t, \boldsymbol{n}, \theta, \boldsymbol{g}, \boldsymbol{x})\cdot\boldsymbol{g}/\theta \geq 0 \, .$$

Equations (31) and (32) reinforce our findings for hyperelastic materials. But (33) shows that the Clausius-Duhem inequality does not suffice to remove $\boldsymbol{n}$ as a constitutive variable.

Let us note that our results with but minor modifications apply to any mechanical theory, such as those of rods, shells, polar media, etc. Indeed, the work of [7] indicates that it can be extended to thermomechanical and even more complicated constraints. Our observations are of potentially great importance in rod and shell theories, which may be placed in hierarchies with the simpler theories representing constrained versions of the more complicated theories. The example treated below gives an inkling of the issues involved.

To illustrate the effect that generalized constitutive relations can have on a well-set problem we describe the buckling of a straight rod of unit reference lenght in a plane. Let $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ be a fixed, right-handed orthonormal basis. The configuration of the rod is specified by a position vector function $\boldsymbol{r}$ from $[0, 1]$ to span $\{\boldsymbol{i}, \boldsymbol{j}\}$, which locates the deformed axis, and a scalar function θ on $[0, 1]$, which gives the orientation of each unit vector $\boldsymbol{a}(s) = \cos\theta(s)\,\boldsymbol{i} + \sin\theta(s)\,\boldsymbol{j}$ normal to the deformed cross-section at s. We set $\boldsymbol{r}'(s) = [1 + \nu(s)]\,\boldsymbol{a}(s) + \eta(s)\,\boldsymbol{k}\times\boldsymbol{a}(s)$. This rod can suffer flexure, extension, and shear. Its strains are ν, η, θ'. (Cf. [2], for a full discussion of this rod theory). Let $\mathrm{N}(s)\,\boldsymbol{a}(s) + \mathrm{H}(s)\,\boldsymbol{b}(s)$ be the resultant contact force and let $\mathrm{M}(s)\,\boldsymbol{k}$ be the resultant contact couple at section s. If the only loads applied to the rod are terminal, then the equilibrium equations are

$$(34) \qquad \frac{d}{ds}\left[N(s)\,\boldsymbol{a}(s) + H(s)\,\boldsymbol{k}\times\boldsymbol{a}(s)\right] = \mathbf{0}\,,$$

$$(35) \qquad M'(s) + (1+\nu)\,H - \eta N = 0\,.$$

The constitutive equations for the unconstrained rod are

$$(36) \qquad N(s) = \hat{N}(\nu(s), \eta(s), \theta'(s), s)\,,\ H(s) = \\ = \hat{H}(\nu(s), \eta(s), \theta'(s), s)\,,\ M(s) = \hat{M}(\nu(s), \eta(s), \theta'(s), s)\,.$$

We assume that the function $(\hat{N}(\cdot\,,\cdot\,,\theta', s)\,\hat{H}(\cdot\,,\cdot\,,\theta', s))$ has an inverse $(\nu^{\#}(\cdot\,,\cdot\,,\theta', s), \eta^{\#}(\cdot\,,\cdot\,,\theta' s))$ so that (36) is equivalent to

$$(37\text{ a, b, c}) \qquad \nu(s) = \nu^{\#}(N(s), H(s), \theta'(s), s)\,,\ \eta(s) = \eta^{\#}(N(s), \\ H(s), \theta'(s), s)\,,\ M(s) = M^{\#}(N(s), H(s), \theta'(s), s)\,.$$

Suppose that $\hat{H}(\nu, 0, \theta', s) = 0$ and $\hat{M}(\nu, \eta, 0, s) = 0$. We study the problem in which

$$(38\text{ a, b, c, d}) \qquad \boldsymbol{r}(0) = \mathbf{0} \quad , \quad \theta(0) = 0\,, \\ N(1)\,\boldsymbol{a}(1) + H(1)\,\boldsymbol{k}\times\boldsymbol{a}(1) = -\lambda\boldsymbol{i} \quad , \quad \theta(1) = 0\,.$$

Then (34) and (38 c) yield explicit formulas for N and H, which we substitute into (35) and (37) to get the following nonlinear eigenvalue problem for θ and λ:

$$(39) \qquad \frac{d}{ds} M^{\#}(-\lambda\cos\theta, \lambda\sin\theta, \theta', s) \\ + \lambda\,[1 + \nu^{\#}(-\lambda\cos\theta, \lambda\sin\theta, \theta', s)]\sin\theta \\ + \lambda\eta^{\#}(-\lambda\cos\theta, \lambda\sin\theta, \theta', s)]\cos\theta = 0\,, \\ \theta(0) = 0 = \theta(1)\,.$$

Next we study the equations for an inextensible, unshearable rod obtained by setting $\nu^{\#} = 0$, $\eta^{\#} = 0$, while retaining (37 c) (in accord with our prescription). Then (30) reduces to

$$(40) \qquad \frac{d}{ds} M^{\#}(-\lambda\cos\theta, \lambda\sin\theta, \theta', s) + \lambda\sin\theta = 0\,,\ \theta(0) = 0 = \theta(1)\,.$$

Had we adhered to the traditional doctrine of replacing (37 c) with $M(s) = M^{\#}(\theta'(s), s)$, then in place of (40) we would obtain

$$(41) \qquad \frac{d}{ds} M^{\#}(\theta', s) + \lambda\sin\theta = 0 \quad , \quad \theta(0) = 0 = \theta(1)\,.$$

Now the linearizations of (40) and (41) about the trivial solution $\theta = 0$ are

$$\frac{d}{ds}\{[\partial M^{\#}(-\lambda, 0, 0, s)/\partial\theta'] \psi'\} + \lambda\psi = 0 \quad , \quad \psi(0) = 0 = \psi(1) , \tag{42}$$

$$\frac{d}{ds}\{\partial M^{\#}(0, s)/\partial\theta'] \psi'\} + \lambda\psi = 0 \quad , \quad \psi(0) = 0 = \psi(1) . \tag{43}$$

The difference between these problems is substantial. Suppose that $\partial M^{\#}(0, s)/\partial\theta' > 0$. Then the Sturmian Theory implies that (43) has a countable infinity of simple positive eigenvalues $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ and corresponding eigenfunctions $\psi_0, \psi_1, \psi_2, \cdots$ with $\lambda_k \to \infty$ as $k \to \infty$ and with ψ_k having exactly k interior zeros on $[0, 1]$, each of which is simple. No such comprehensive statement is possible for (42). Indeed, (42) may have just a finite of eigenvalues (cf. [2]). Thus (40) has much of the richness of response of the unconstrained problem (39) so that the qualitative behavior of its solutions, which can be completely determined (cf. [2]), may differ markedly from that of the solutions of (41).

This example illustrates that there can be significant differences between the behavior of constrained hyperelastic materials and that of constrained non-hyperelastic materials. For the latter, many of the expected simplifications need not be present when our more general constitutive functions are used. We note that these differences are even more pronounced in various problems for visco-elastic materials, where the artificial distinction between hyperelasticity and Cauchy elasticity does not intervene.

We finally observe that a small parameter ε can be introduced into constitutive equations (9) and (10) so that $v^{\#} = 0$ when $\varepsilon = 0$. In this case we could readily construct a perturbation scheme to yield representations of solutions of boundary value problems for nearly constrained materials. The construction of such schemes for constitutive equations in the form (7) and (8) is somewhat more difficult and by no means unique. Cf. [1], [3] and [8]. A rigorous treatment of such a perturbation scheme has been given by Ebin [4].

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Note added in proof. A. Needleman has pointed out to me that constitutive equations of the form (6) for incompressible media are used to describe important phenomena in various theories of plasticity. Cf. [10].