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The bounce problem, on n-dimensional Riemannian manifolds

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Riassunto. — In questo lavoro vengono generalizzati i risultati relativi al problema del rimbalzo unidimensionale studiato in [5]. Precisamente si considera un punto mobile su una varietà Riemanniana V n-dimensionale, soggetto all'azione di un potenziale variabile nel tempo e vincolato a restare in una parte W di V avente un bordo di classe C3 contro cui il punto «rimbalza».

Lo studio del problema richiede l'uso di metodi di l'-convergenza del tipo usato in [5], metodi che sembrano caratteristici per lo studio di problemi in cui può mancare l'unicità della soluzione o la sua dipendenza continua dai dati.

Let V be an n-dimensional Riemannian manifold without boundary and of class C3; for every q ∈ V we denote by ( , )q the scalar product on the tangent space Tq V; we assume that the metric tensor field is of class C2.

If g : V → ℝ is a differentiable function, we denote by ∇g the gradient of g defined by

\[ (\nabla g(q), v)_q = dg(q)(v) \]

for every v ∈ Tq V.

Let f : V → ℝ be a function of class C3; we suppose that df(q) ≠ 0 for every q ∈ V such that f(q) = 0.

We study the bounce problem for a material point, whose position at time t will be indicated by q(t). This point is subjected to a potential U(t, q) and moves in the region W = \{q ∈ V : f(q) ≥ 0\} bouncing against the submanifold S = \{q ∈ V : f(q) = 0\}.

Given T > 0 we denote by Lip the space of Lipschitz functions from [0, T] into V endowed with the L∞-topology and by L1 the space L1(0, T ; C2(V)) with its usual topology.

(*) Nella seduta del 26 giugno 1981.
We say that a pair \((U, q) \in L^1 \times \text{Lip}\) solves the bounce problem \((\mathcal{P})\) (or that \(q\) is a solution of the bounce problem \((\mathcal{P})\) with potential \(U\)) if

\[
\begin{align*}
&i) \quad f(q(t)) \geq 0 \text{ for every } t \in [0, T] \\
&ii) \text{ there exists a positive measure } \mu \text{ on } (0, T) \text{ such that } q \text{ is an extremal for the functional } \\
&\int_0^T \left[ \frac{1}{2} \langle v'(t), v'(t) \rangle + U(t, v(t)) \right] dt + \int_0^T f(v(t)) d\mu \\
&\text{and } \text{spt } \mu \subseteq \{ t \in [0, T] : f(q(t)) = 0 \} \\
&iii) \text{ for every } t_1, t_2 \in [0, T] \text{ the energy-relation holds: }
\end{align*}
\]

where \(q^+\) and \(q^−\) respectively denote the right and left derivatives of \(q\), whose existence is guaranteed by \(\text{ii)}\). By \(\text{iii)}\) the functions \(\{q^+, q^+_t\}\) and \(\{q^−, q^−_t\}\) coincide; their common value will be indicated by \(\{q^+, q^−\}\).

We consider now a sequence of penalizing functions \(\psi_h\) satisfying the following properties:

\[
\begin{align*}
i) & \quad \psi_h \text{ is continuous, } \psi_h \geq 0, \quad \psi_h(x) = 0 \text{ if } x \geq 0 \\
ii) & \quad \psi_h \to +\infty \text{ uniformly on compact subsets of } (-\infty, 0) \\
iii) & \quad \lim_{h \to +\infty} \frac{\psi_h(x)}{\alpha_h(x)} = +\infty \text{ where } \alpha_h(x) = \int_0^x \psi_h(z) dz .
\end{align*}
\]

We say that a pair \((U, q) \in L^1 \times \text{Lip}\) solves the penalized problem \((\mathcal{P}_h)\) (or that \(q\) is a solution of \((\mathcal{P}_h)\) with potential \(U\)) if \(q\) is an extremal for the functional

\[
\Gamma_h(v) = \int_0^T \left[ \frac{1}{2} \langle v'(t), v'(t) \rangle + U(t, v(t)) - \alpha_h(f(v(t))) \right] dt .
\]

**Remark.** It is easy to see that if \((U, q)\) satisfies \((\mathcal{P})\) \(\text{ii)}\) then, in local coordinates we have (the summation convention is adopted):

\[
\frac{d}{dt} (a_{ij}(q) q_j') = \frac{\partial}{\partial q_i} \left[ \frac{1}{2} a_{ij}(q) q_i' q_j' + U(t, q) \right] + \mu \frac{\partial f}{\partial q_i}(q)
\]

where \(a_{ij}\) denote the coefficients of the metric tensor.
Similarly, if \((U, q)\) solves the problem \((\mathcal{P}_h)\) then, in local coordinates we have
\[
\frac{d}{dt} (a_{ij}(q) q'_i) = \frac{\partial}{\partial q_l} \left[ \frac{1}{2} a_{rs}(q) q'_r q'_s + U(t, q) - \alpha_h(f(q)) \right].
\]

We are interested in the Cauchy problems for \((\mathcal{P}_h)\) and \((\mathcal{P})\); nevertheless we cannot assign the usual Cauchy data since they are not stable as \(h \to +\infty\). In fact, if \(U \in L^1\) is a potential, \(q_h\) solves \((\mathcal{P}_h)\) with potential \(U\), \(q\) solves \((\mathcal{P})\) with potential \(U\) and \(q_h \to q\) uniformly, then the usual Cauchy data of the problem \((\mathcal{P}_h)\) may not be convergent to the Cauchy data of the problem \((\mathcal{P})\). Moreover the usual Cauchy data for the problem \((\mathcal{P})\) are discontinuous as functions of time \(t\). These facts lead us to introduce a new kind of "initial trace" for \((\mathcal{P}_h)\) and \((\mathcal{P})\), which is more stable than the usual one as \(h \to +\infty\).

Let
\[
\mathcal{E}_h = \{(U, q) \in L^1 \times Lip : (U, q) \text{ solves } \mathcal{P}_h\}
\]
\[
\mathcal{E} = \{(U, q) \in L^1 \times Lip : (U, q) \text{ solves } \mathcal{P}\}
\]
\[
\mathcal{Y}_h = \{q \in Lip : \exists U \in L^1 \text{ with } (U, q) \in \mathcal{E}_h\}
\]
\[
\mathcal{Y} = \{q \in Lip : \exists U \in L^1 \text{ with } (U, q) \in \mathcal{E}\}
\]
setting \(\mathcal{B} = \mathbb{R} \times V \times TV \times TV \times TV\) we define the "initial traces"
\[
\mathcal{C}_h : [0, T] \times \mathcal{Y}_h \to \mathcal{B} \text{ and } \mathcal{C} : [0, T] \times \mathcal{Y} \to \mathcal{B} \text{ by}
\]
\[
\mathcal{C}(t, q) = (\mathcal{e}_h(q)(t), q(t), q'_r(t), f(q(t)) q'(t), \frac{1}{h} q'(t))
\]
\[
\mathcal{C}(t, q) = (\mathcal{e}(q)(t), q(t), q'(t), q'_r(t), f(q(t)) q'(t), 0)
\]
where \(q'_r\) denotes the vector \(\langle \nabla f(q) , \nabla f(q) \rangle q' - (q' , \nabla f(q)) \nabla f(q)\), and \(\mathcal{e}_h(q), \mathcal{e}(q)\) are respectively the energies \(\frac{1}{2} \langle q' , q' \rangle + \alpha_h(f(q))\) and \(\frac{1}{2} \langle q' , q' \rangle\).

It is easy to verify that \(\mathcal{C}_h, \mathcal{C}\) are continuous with respect to \(t\) for every fixed \(q\) in \(\mathcal{Y}_h, \mathcal{Y}\) respectively.

We remark that, if \(f(q(t_0)) > 0\), then to assign \(\mathcal{C}(t_0, q)\) is equivalent to assigning the Cauchy data \(q(t_0), q'_r(t_0)\) for the problem \((\mathcal{P})\); to assign \(\mathcal{C}_h(t_0, q)\) is, for the problem \((\mathcal{P}_h)\), always equivalent to assigning the usual Cauchy data.

We set for every \(t \in [0, T]\)
\[
\mathcal{A}_h(t) = \{(b, U, q) \in \mathcal{B} \times L^1 \times Lip : (U, q) \in \mathcal{E}_h, \mathcal{C}_h(t, q) = b\}
\]
\[
\mathcal{A}(t) = \{(b, U, q) \in \mathcal{B} \times L^1 \times Lip : (Y, q) \in \mathcal{E}, \mathcal{C}(t, q) = b\};
\]
we notice that, for fixed \( t \in [0, T] \), the relation \( \mathcal{A}(t) \) does not uniquely characterize \( q \) as a function of \( (b, U) \) (see [6]). Failing uniqueness for the problem \( (\mathcal{P}) \) with fixed initial conditions, the following question naturally arises:

may every solution of the problem \( (\mathcal{P}) \) be obtained as a limit of solutions of problems \( (\mathcal{P}_h) \) ?

We answer this question in the affirmative by allowing a suitable "mobility" of potential and of initial conditions. To do this we use the notions of \( \Gamma \)-limits.

If \( X \) is a set and \( M \subseteq X \) we define

\[
\delta_M(x) = \begin{cases} 
0 & \text{if } x \in M \\
\infty & \text{otherwise};
\end{cases}
\]

we recall (see [4], [8], [9]) that we say

\[
\delta_{\mathcal{A}(0)} = \Gamma \left( N^-, \mathcal{B}^-, (L^1)^-, (L^\infty)^- \right) \lim_{h} \delta_{\mathcal{A}_h(0)}
\]

if and only if the following properties hold:

I) If \( b_h \to b \) in \( \mathcal{B}, U_h \to U \) in \( L^1 \), \( q_h \to q \) in \( L^\infty \) with \( (b_h, U_h, q_h) \in \mathcal{A}_h(t) \) for infinitely many \( h \in N \), then \( (b, U, q) \in \mathcal{A}(t) \).

II) If \( (b, U, q) \in \mathcal{A}(t) \), then there exist \( b_h \to b \) in \( \mathcal{B}, U_h \to U \) in \( L^1 \), \( q_h \to q \) in \( L^\infty \) such that \( (b_h, U_h, q_h) \in \mathcal{A}_h(t) \) for all \( h \) large enough.

Our purpose is to obtain the following result:

**Theorem 1.** For every \( t \in [0, T] \) we have

\[
\delta_{\mathcal{A}(0)} = \Gamma \left( N^-, \mathcal{B}^-, (L^1)^-, (L^\infty)^- \right) \lim_{h} \delta_{\mathcal{A}_h(0)}.
\]

The proof of the Theorem 1 mainly consists in the following steps:

**Proposition 2.** For every \( t \in [0, T] \)

\[
\delta_{\mathcal{A}(0)} \leq \Gamma \left( N^-, \mathcal{B}^-, (L^1)^-, (L^\infty)^- \right) \lim_{h} \delta_{\mathcal{A}_h(0)}.
\]

This Proposition proves property I). In order to prove property II), we need
the notion of regular solution for the problem \( (\mathcal{P}) \);

**Definition 3.** We set for every \( t \in [0, T] \)

\[
D(t) = \{(b, U, q) \in \mathcal{A}(t) : \forall \tilde{q} \in \text{Lip}, (b, U, \tilde{q}) \in \mathcal{A}(t) \Rightarrow q = \tilde{q}\}.
\]

We call \( D(t) \) the set of regular solutions of problem \( (\mathcal{P}) \) with initial time \( t \).
It is easy to obtain the following

**Proposition 4.** For every $t \in [0, T]$

$$\delta_{D(t)} \geq \Gamma (N^+, B^+, (L^1)^+, (L^\infty)^-) \lim_{h} \delta_{\mathcal{A}(t)}$$

that is, if $(b, U, q) \in D(t)$ then, for every $b_h \to b$ in $B$, for every $U_h \to U$ in $L^1$ there exists $q_h \to q$ in $L^\infty$ such that $(b_h, U_h, q_h) \in st_{\mathcal{A}(t)}$ for all $h$ large enough.

The crucial point in the proof of property II) consists in proving the following

**Theorem 5.** For every $t \in [0, T]$ $D(t)$ is dense in $\mathcal{A}(t)$ with respect to the product topology of $B \times L^1 \times L^\infty$.

The thesis of Theorem 1 follows now from the previous results and from general $\Gamma$-convergence results (see [4], [8], [9]).

**References**


