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Quasi-completeness on the Spaces of Holomorphic Germs

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Riassunto. — Sia E uno spazio DF riflessivo e sia K un compatto di E. Si dimostra che lo spazio dei germi olomorfi su K, con la topologia naturale, è un limite induttivo regolare e quasi completo purché lo spazio dei germi olomorfi all'origine sia un limite induttivo regolare.

Let E be a complete Hausdorff, complex, locally convex space and let K be a compact subset of E. Let \( \left( H^\omega(U) ; \| \| \right) \) denote the Banach space of all bounded holomorphic functions on the open subset U of E with the supremum norm. We denote by \( H(K) \) the space of holomorphic germs on K endowed with its natural topology defined as follows

\[
H(K) = \lim_{U \supseteq K} (H^\omega(U) ; \| \|)
\]

where U runs over the collection of all open subsets of E which contain K. We say that \( H(K) \) is regular if any bounded subset B of \( H(K) \) is contained and bounded in some \( H^\omega(U) \). We refer to [2] and [3] for basic information in \( H(K) \) and [7] for the study of regularity of \( H(0) \). Let B be a bounded subset of \( H(K) \). As pointed out in [4], regularity of \( H(0) \) implies the existence of

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an open neighbourhood $V$ of zero in $E$ such that for any $x \in K$ and $f \in B$ we can define 

\[ \tilde{f}(x)(y) = \sum_{m=0}^{\infty} \frac{(1/m!)}{m!} \tilde{d}^m f(x)(y) \]

for $y \in V$. If such Taylor series expansions are coherent (i.e. if there exists an open neighbourhood of zero $W \subset V$ such that $\tilde{f}(x)(y) = \tilde{f}(x')(y')$ for all $x, x' \in K$, $y, y' \in K$, $f \in B$ whenever $x + y = x' + y'$), then $B$ is contained and bounded in $H^\infty(K + W)$. Hence the study of regularity of $H(K)$ can be reduced to the following

**Question 1.** When does regularity of $H(0)$ imply coherence of the Taylor series expansion of the elements of a bounded subset of $H(K)$?

Such a question has a positive answer if we restrict ourselves to locally convex spaces satisfying the following technical condition.

**Definition 2.** (As [7]) We say that the locally convex space $E$ satisfies condition $P$ if for each convex, balanced, open subset $U$ of $E$ and for any sequence $(f_n)_{n=0}^{\infty}$ of non-zero holomorphic functions on $U$, there exist a subsequence $(f_{n_j})_{j=0}^{\infty}$ and a bounded sequence $(z_j)_{j=0}^{\infty}$ in $E$ such that $f_{n_j}(z_j) \neq 0$ for all $j \in \mathbb{N}$.

If the answer to question 1 is negative, for the compact metrizable subset $K$ of $E$ we can find sequences $(x_n)_{n=0}^{\infty}$ and $(x'_n)_{n=0}^{\infty}$ in $K$, a sequence $(f_n)_{n=0}^{\infty}$ in a bounded subset $B$ of $H(K)$ such that the sequence $(x_n - x'_n)_{n=0}^{\infty}$ is a null sequence and $h_n(y) = f_n(x_n)(y) - \tilde{f}_n(x_n)(x_n - x'_n + y)$ defines a sequence of non-zero holomorphic functions on a convex, balanced, open neighbourhood of zero in $E$. Definition 2 is applied to get a bounded sequence $(z_j)_{j=0}^{\infty}$ and a subsequence $(h_{n_j})_{j=0}^{\infty}$ such that $h_{n_j}(z_j) \neq 0$ for all $j \in \mathbb{N}$. We use the null sequence $(x_n - x'_n)_{n=0}^{\infty}$ and the bounded sequence $(z_j)_{j=0}^{\infty}$ to construct a continuous seminorm on $H(K)$ which is not bounded on the bounded subset $B$. This gives the required contradiction and we have the following result.

**Theorem 3.** Let $E$ be a locally convex space satisfying condition $P$. Then $H(K)$ is regular whenever $H(0)$ is regular and $K$ is a metrizable compact subset of $E$.

Baire and metrizable locally convex spaces as well as any product of metrizable locally convex spaces are examples of spaces satisfying condition $P$. To get further examples, we need another characterization of condition $P$.

**Definition 4.** (As [5]) The sequence $(x_n)_{n=0}^{\infty}$ in $E$ is a very strongly convergent sequence if for any sequence of scalars $(\lambda_n)_{n=0}^{\infty}$ the sequence $(\lambda_n x_n)_{n=0}^{\infty}$ is a null sequence in $E$. The sequence $(x_n)_{n=0}^{\infty}$ is non-trivial if $x_n \neq 0$ for all $n$.

**Proposition 5.** The complete locally convex space $E$ satisfies condition $P$ if and only if there is no non-trivial very strongly convergent sequence in $H(E)$ (where the topology of $H(E)$ is the compact open topology $\tau_0$).
If $E$ is a reflexive DF space, proposition 5 implies that $E$ satisfies condition P if and only if the strong dual $E^{\#}$ has a continuous norm. On the other hand, it is known, [7], that if $F$ is a Fréchet space without continuous norm, then $H(0), 0 \in F^{\#}$ is not regular. Hence we get

**Proposition 6.** Let $E$ be a reflexive DF space. If $H(0), 0 \in E$, is regular, then $E$ satisfies condition P.

Now, recalling that every compact subset of a DF space is metrizable ([6] or [8]) and barrelled DF spaces are quasi-normable (in this case regularity of $H(K)$ implies quasi-completeness of $H(K)$, [1] or [2]), theorem 3 and proposition 6 imply

**Theorem 7.** Let $E$ be a reflexive DF space. For any compact subset $K$ of $E$, $H(K)$ is a quasi-complete and regular inductive limit whenever $H(0), 0 \in E$, is regular.

**References**


