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Quasi-completeness on the Spaces of Holomorphic Germs

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Quasi-completeness on the Spaces of Holomorphic Germs.* Nota di ROBERTO LUIZ SORAGGI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Sia E uno spazio DF riflessivo e sia K un compatto di E. Si dimostra che lo spazio dei germi olomorfi su K, con la topologia naturale, è un limite induttivo regolare e *quasi* completo purché lo spazio dei germi olomorfi all'origine sia un limite induttivo regolare.

Let E be a complete Hausdorff, complex, locally convex space and let K be a compact subset of E. Let $(H^{\infty}(U) ; \| \|)$ denote the Banach space of all bounded holomorphic functions on the open subset U of E with the supremum norm. We denote by H (K) the space of holomorphic germs on K endowed with its natural topology defined as follows

$$H(K) = \varinjlim_{U \supset K} (H^{\infty}(U) ; \| \ \|)$$

where U runs over the collection of all open subsets of E which contain K. We say that H (K) is regular if any bounded subset B of H (K) is contained and bounded in some $H^{\infty}(U)$. We refer to [2] and [3] for basic information in H (K) and [7] for the study of regularity of H (0). Let B be a bounded subset of H (K). As pointed out in [4], regularity of H (0) implies the existence of

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an open neighbourhood V of zero in E such that for any $x \in$ K and $f \in$ B we can define $\tilde{f}(x)(y) = \sum_{m=0}^{\infty} (1/m!) \hat{d}^m f(x)(y)$ for $y \in$ V. If such Taylor series expansions are coherent (i.e. if there exists an open neighbourhood of zero $W \subset V$ such that $\tilde{f}(x)(y) = \tilde{f}(x')(y')$ for all $x, x' \in K$, $y, y' \in K$, $f \in B$ whenever $x + y = x' + y'$), then B is contained and bounded in $H^\infty(K + W)$. Hence the study of regularity of H (K) can be reduced to the following

Question 1. When does regularity of H (0) imply coherence of the Taylor series expansion of the elements of a bounded subset of H (K) ?

Such a question has a positive answer if we restrict ourselves to locally convex spaces satisfying the following technical condition.

DEFINITION 2. (As [7]) We say that the locally convex space E satisfies condition P if for each convex, balanced, open subset U of E and for any sequence $(f_n)_{n=0}^{\infty}$ of non-zero holomorphic functions on U, there exist a subsequence $(f_{n_j})_{j=0}^{\infty}$ and a bounded sequence $(z_j)_{j=0}^{\infty}$ in E such that $f_{n_j}(z_j) \neq 0$ for all $j \in \mathbf{N}$.

If the answer to question 1 is negative, for the compact metrizable subset K of E we can find sequences $(x_n)_{n=0}^{\infty}$ and $(x_n')_{n=0}^{\infty}$ in K, a sequence $(f_n)_{n=0}^{\infty}$ in a bounded subset B of H (K) such that the sequence $(x_n - x_n')_{n=0}^{\infty}$ is a null sequence and $h_n(y) = \tilde{f}_n(x_n)(y) - \tilde{f}_n(x_n')(x_n - x_n' + y)$ defines a sequence of non-zero holomorphic functions on a convex balanced, open neighbourhood of zero in E. Definition 2 is applied to get a bounded sequence $(z_j)_{j=0}^{\infty}$ and a subsequence $(h_{n_j})_{j=0}^{\infty}$ such that $h_{n_j}(z_j) \neq 0$ for all $j \in \mathbf{N}$. We use the null sequence $(x_n - x_n')_{n=0}^{\infty}$ and the bounded sequence $(z_j)_{j=0}^{\infty}$ to construct a continuous seminorm on H (K) which is not bounded on the bounded subset B. This gives the required contradiction and we have the following result.

THEOREM 3. *Let* E *be a locally convex space satisfying condition* P. *Then* H (K) *is regular whenever* H (0) *is regular and* K *is a metrizable compact subset of* E.

Baire and metrizable locally convex spaces as well as any product of metrizable locally convex spaces are examples of spaces satisfying condition P. To get further examples, we need another characterization of condition P.

DEFINITION 4. (As [5]) The sequence $(x_n)_{n=0}^{\infty}$ in E is a very strongly convergent sequence if for any sequence of scalars $(\lambda_n)_{n=0}^{\infty}$ the sequence $(\lambda_n x_n)_{n=0}^{\infty}$ is a null sequence in E. The sequence $(x_n)_{n=0}^{\infty}$ is non-trivial if $x_n \neq 0$ for all n.

PROPOSITION 5. *The complete locally convex space* E *satisfies condition* P *if and only if there is no non-trivial very strongly convergent sequence in* H (E) (*where the topology of* H (E) *is the compact open topology* τ_0).

If E is a reflexive DF space, proposition 5 implies that E satisfies condition P if and only if the strong dual E'_β has a continuous norm. On the other hand, it is known, [7], that if F is a Fréchet space without continuous norm, then H (0), $0 \in F'_\beta$ is not regular. Hence we get

PROPOSITION 6. *Let* E *be a reflexive* DF *space. If* H (0) , $0 \in E$, *is regular, then* E *satisfies condition* P.

Now, recalling that every compact subset of a DF space is metrizable ([6] or [8]) and barrelled DF spaces are quasi-normable (in this case regularity of H (K) implies quasi-completeness of H (K), [1] or [2]), theorem 3 and proposition 6 imply

THEOREM 7. *Let* E *be a reflexive* DF *space. For any compact subset* K *of* E, H (K) *is a quasi-complete and regular inductive limit whenever* H (0), $0 \in E$, *is regular.*

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