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Cylindrical real hyper surfaces in $\mathbf{C}^n$

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Geometria differenziale. — *Cylindrical real hypersurfaces in* $\mathbf{C}^n$ (*).
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RIASSUNTO. — Si stabiliscono due condizioni sufficienti per un germe di ipersuperficie reale di classe C^∞ in $\mathbf{C}^n$, affinchè esistano coordinate olomorfe rispetto alle quali l'ipersuperficie risulti essere il luogo di zeri di una funzione di $k < n$ variabili e k sia minimale rispetto a questa proprietà. In altre parole si vuole che l'ipersuperficie, a meno di una trasformazione bi-olomorfa, sia l'unione di sottovarietà lineari complesse, parallele di dimensione $n - k$.

§ 1. A common problem in complex analysis is to ask whether a real, smooth hypersurface $S \subset \mathbf{C}^n$ of equation

$$F(z_1, \cdots, z_n) = 0$$

can be written in the form

$$\phi(\zeta_1, \cdots, \zeta_k) = 0\,, \qquad \text{with } k < n \tag{1}$$

by a suitable choice of the holomorphic coordinates $\zeta_1, \cdots, \zeta_n$ at some point $p \in S$. If k is as small as possible we say that S is a *irreducible* $(n - k)$-*cylinder.* Scope of the present paper is to give a partial answer to this question when $k \geq 2$.

Let L be the Levi form of S restricted to the complex tangent space $T = \left\{u = \sum_1^n{}_j\, u_j\, \partial_j \mid uF = 0\right\}$ of S, and N the kernel of L as linear map on T.

THEOREM 1. *Let the hypersurface* S *satisfy the following two conditions for* $k \geq 2$

$$rk L_q = k - 1\,, \qquad \textit{for all } q \textit{ near } p \tag{2}$$

$$\sum_1^n{}_{jkl} (F_{j\bar{h}\bar{l}}\, F_{\bar{s}} - F_{j\bar{s}}\, F_{\bar{h}\bar{l}})\, v_j\, \bar{u}_l\, \bar{w}_h = 0\,, \qquad (s = l, \cdots, n) \tag{3}$$

for all $v \in N$, $u, w \in T$,
then

(i) *if* S *is not pseudoconvex, then* S *is an irreducible* $(n - k)$-*cylinder,*

(ii) *if* S *is pseudoconvex, say* $L \geq 0$, *then there are* C^∞ *complex coordinates* $\zeta_1, \cdots, \zeta_n$ *in a neighbourhood* U *of* p, *holomorphic on the pseudoconvex side*

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U^- *of* U, *such that* S *has equation* (1) *in* U. *In particular* S *is the limit* (*uniform with all derivatives*) *of irreducible* $(n-k)$*-cylinders* $S_\varepsilon \equiv \{\phi = \varepsilon\}$, *for* $\varepsilon \to 0^-$.

Only subscripts to F denote derivations.

The *cross section* of a cylinder S is the hypersurface S^0 of $\mathbf{C}^k_\zeta$ having (1) as equation. S^0 is unique up to a bi-holomorphism.

Equation (3) may seem strange. Its meaning is explained in proposition 1, anyway (3) is necessary in the sense of the following

Necessary condition. *Let* S *be a* $(n-k)$*-cylinder at* p. *Then condition* (2) *is equivalent to the nondegeneracy of the Levi form of the cross section* S^0 *and equation* (3) *is a consequence of* (2).

In view of part (ii) we construct in § 3 a family of complex lines in $\mathbf{C}^3$ whose union is a pseudoconvex hypersurface which satisfies (2) and (3) without being a cylinder, and the tangent space of the generating lines is the Levi kernel N.

Before considering further results, let us point out which cylinders are *not* recognizable with Theorem 1.

An irreducible $(n-k)$-cylinder escapes Theorem 1 if

1) S is generated by complex hyperplanes, i.e. $k=1$,

2) S^0 has degenerate Levi form at p.

3) Finally, if S^0 is strictly pseudoconvex at p, then Theorem 1 (ii) only detects that S is a limit of irreducible $(n-k)$-cylinders.

For inspecting those remaining cases, consider the module V of C^∞ complex vector fields tangent to the generating lines of a $(n-k)$-cylinder. V satisfies trivially the conditions

$$[V, V] \subset V, \ [V, \overline{T}]^{10} \subset V. \quad (4)$$ [1]

Next theorem affirms that, under suitable hypothesis, the existence of such a module V is also sufficient.

THEOREM 1′. *Assume that* S *admits a module* V *of* C^∞ *complex tangent vector fields of constant dimension* $n-k$ *near* p, *with* $k \geq 2$, *fulfilling* (4), *then the conclusions* (i) *and* (ii) *of theorem* 1 *hold.*

By the argument above only $(n-1)$-cylinders escape this theorem. However this result is much weaker than Theorem 1 because the hypothesis are not directly verifiable on the equation of the surface.

(1) Modules of C^∞ vector fields with this property are called by Freeman ([5]) straight modules. We will add the requirement that V has constant dimension.

Let us explain the meaning of conditions (3) and (4). If the rank of L is constantly equal to $k-1$, as condition (2) prescribes, then $q \mapsto N_q$ is a smooth map of S into the complex Grassmannian $G(n-k, n)$, call this map the *Levi-Gauss map.*

A C^1 function f on S is said to be a CR function if there is $\sum_{1}^{n} {}_j u_j \bar{\partial}_j f = 0$, for all $u \in T$. A CR map is a map whose components are CR functions.

PROPOSITION 1. *If the rank of the Levi form is constant, then* (3) *is equivalent to the property of the Levi-Gauss map to be a* CR*-map.*

In other words (3) means that N is a CR-bundle over S. Similarly Bedford and Kalka have proved in [3] that N is the largest involutive sub-bundle of T which is holomorphic with respect to its own foliation. Here also one can say that condition (3) implies that N is the largest sub-bundle of T which is a CR-bundle.

Let $\mathcal{N}$ be the real representative of N, i.e. $\mathcal{N} \equiv \{v + \bar{v},\ \text{with } v \in N\}$. There is $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$ ([9]), so $\mathcal{N}$ is the tangent space of a foliation of S wherever L has constant rank. This is the *Levi foliation*; its leaves are complex manifolds.

Similarly, let $\mathcal{V}$ be the real representative of V. Then (4) implies obviously $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$. On the other hand, if $v \in V$ and $u \in T$ are vector fields, there is $[v, \bar{u}]^{10} F = \bar{u}(vF) - \sum_{1}^{n} {}_{jh} F_{j\bar{h}} v_j \bar{u}_h$, therefore $(4)_2$ implies that V is in the Levi kernel N. Hence V is the complex tangent space of a complex sub-foliation of the Levi foliation when the latter exists.

PROPOSITION 1′. *Let* V *be any module of* C^∞ *complex vector fields, of constant dimension* $n-k$ *near* p*, tangent to* S. *Condition* $(4)_2$ *implies that the map* $z \mapsto V_z$ *of* S *into the complex grassmannian* $G(n-k, n)$ *is a* CR *map, and vice versa.*

For $k = 1$ (Levi-flat S) condition (3) is void and the problem of characterizing smooth $(n-1)$-cylinders is somehow related to the search of sufficient conditions for the holomorphic extension of a C^∞ function from a C^∞ curve in $\mathbf{C}$. So there are few hopes for that. In the real analytic case all Levi flat submanifolds of any dimension are cylinders over a real curve. This is proved in [8] and Bedford improved the result for singular S ([2]). In [8] there is also an example of a 1-disk moving in $\mathbf{C}^2$ by a composition of a translation with an oscillating motion having diverging frequence and rapidly decreasing amplitude; the disk describes a Levi-flat hypersurface which is not a cylinder.

Theorem 1, in the real analytic case, is known for any k and any codimension of S ([5]). There condition (3) is replaced by an equivalent one similar to our (4), nonpseudoconvexity is unnecessary for the real analytic case. In [4] an example is shown of a cubic hypersurface which satisfies (2) but is not a cylinder for not having fulfilled also (3). For a characterization of *complex* hypersurfaces which are cylinders see [1].

§ 2. *Proofs.*

LEMMA. *Conditions* (2) *and* (3) *imply that* N *satisfies* (4).

Proof. $(4)_1$ is a consequence of $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, so we must only prove

$$\sum_{1}^{n}{}_{jh} [v, \bar{u}]_j^{10} F_{j\bar{h}} \bar{w}_h = 0 \tag{5}$$

for C^∞ vector fields $u, w \in T, v \in N$.

Assume $F_n = 1$ on S and set $c = \sum_1^n{}_j v_j F_{j\bar{n}}$. Since $v \in N$, there is $\sum_1^n{}_j v_j F_{j\bar{h}} = cF_{\bar{h}}$ for all $h \leq n$. Application of the operator $\sum_1^n{}_l \bar{u}_l \bar{\partial}_l$ to both sides and multiplication by $\bar{w}_h$ give

$$\sum_{1}^{n}{}_{jk} [v, u]_j^{10} F_{j\bar{h}} \bar{w}_h = \sum_{1}^{n}{}_{jhl} F_{j\bar{h}\bar{l}} v_j \bar{u}_l \bar{w}_h - c \sum_{1}^{n}{}_{hl} F_{\bar{h}\bar{l}} \bar{u}_l \bar{w}_h .$$

If we take $s = n$ in (3), we have the desired result.

COROLLARY. *Theorem* 1′ *and proposition* 1′ *imply theorem* 1 *and proposition* 1 *respectively.*

Proof. Take V = N and apply the lemma above.

In the next proofs we take coordinates with origin at p, *such that the* $(z_1, \cdots, z_k)$*-plane is transversal to* V *at* p, *and set* $z' = (z_1, \cdots, z_k)$, $z'' = (z_{k+1}, \cdots, z_n)$.

Proof of proposition 1′. For z^0 near p, $V(z^0)$ has equations

$$z_A - z_A^0 = \sum_{k+1}^{n}{}_a V_{aA}(z^0)(z_a - z_a^0), \qquad A = 1, \cdots, k.$$

We must prove

$$\sum_{1}^{n}{}_j \bar{u}_j \bar{\partial}_j V_{aA} = 0, \qquad \text{for all } 1 \leq A \leq k < a \leq n, u \in T. \tag{6}$$

The vectors $e_a = \partial_a + \sum_1^k{}_A V_{aA} \partial_A$ are a basis for $V (a = k + 1, \cdots, n)$. For all vector fields $v \in V, u \in T$ of class C^∞, we have by $(4)_2$

$$\sum_{1}^{n}{}_j \bar{u}_j (\bar{\partial}_j v_h) \partial_h = \sum_{k+1}^{n}{}_a c_a e_a .$$

Replace here v by e_b $(k+1 \leq b \leq n)$. Taking the last $n-k$ components of both sides we obtain $c_{k+1} = \cdots = c_n = 0$. The consequent vanishing of the first k components of the left side gives (6). Viceversa (6) implies $[e_a, \bar{u}]^{10} = 0$, for all $u \in \mathrm{T}$ and $a = k+1, \cdots, n$, i.e. $(4)_2$.

Proof of theorem 1′. In both cases (*i*) and (*ii*) L is non degenerate at p, so, by (6), H. Lewy Theorem ([7] or [6] th. 2.6.13) applies to the restrictions to S of the functions $\mathrm{V}_{a\mathrm{A}}$ and there exists a neighbourhood U of p such that those functions extend holomorphically to $\mathrm{U}^- = \{z \in \mathrm{U} \mid \mathrm{F}(z) < 0\}$, C^∞ on $\overline{\mathrm{U}}^-$. In the case (i) U^- is not pseudoconvex, so $\mathrm{V}_{a\mathrm{A}}$ can be furtherly continued to U (shrunk if necessary) past p. The new holomorphic functions $\mathrm{V}_{a\mathrm{A}}$ satisfy by $(4)_1$ the identity

$$\partial_a \mathrm{V}_{b\mathrm{A}} - \partial_b \mathrm{V}_{a\mathrm{A}} + \sum_1^k {}_{\mathrm{B}} \mathrm{V}_{a\mathrm{B}} (\partial_{\mathrm{B}} \mathrm{V}_{b\mathrm{A}} - \partial_{\mathrm{B}} \mathrm{V}_{a\mathrm{A}}) = 0, \quad \text{for } 1 \leq \mathrm{A} \leq k < a, b \leq n,$$

on S, hence on U.

Hence, by the holomorphic version of Frobenius classical theorem, there is a holomorphic vector function $h = (h_{k+1}, \cdots, h_n)$ on U such that

$$\partial_a h_{\mathrm{A}}(z) = \mathrm{V}_{a\mathrm{A}}[h(z), z''], \quad \text{for } 1 \leq \mathrm{A} \leq k < a \leq n, \quad \text{and} \quad h(z', 0) = z'.$$

The coordinates ζ in U defined by $z' = h(\zeta', z'')$, $z'' = \zeta''$ give S the equation (1). For the second part we only need to note that the first one-sided holomorphic extension $\mathrm{V}_{a\mathrm{A}}$ can be taken of class C^∞ on U.

Q.E.D.

Note that, for real analytic S, the CR condition (6) on $\mathrm{V}_{a\mathrm{A}}$ allows to extend holomorphically those functions indipendently of the non convexity of S and of $k \geq 2$. This proves the mentioned Freeman's theorem in the case of hypersurfaces.

§ 3. *Example*. We want to show a pseudoconvex limit of 1-cylinders in $\mathbf{C}^3$, satisfying the hypothesis of Theorem 1 (ii), which is not a cylinder. Let $\lambda(z_1)$ be holomorphic on $|z_1| < 1$, C^∞ up to the boundary, not holomorphycally extendable past $z_1 = 1$. Consider the family of complex lines.

$$\mathscr{F} \begin{cases} z_1 = z_1^0 + \lambda(z_1^0)\, z_3 \overset{\mathrm{def}}{=} h_1(z^0, z_3) \\ z_2 = z_2^0 + z_3 \overset{\mathrm{def}}{=} h_2(z^0, z_3) \end{cases} \qquad z^0 = (z_1^0, z_2^0), \; |z^0| = 1.$$

This family describes a hypersurface $\mathrm{S} \subset \mathbf{C}^3$ which will be studied near the point $p = (1, 0, 0)$. If $z(\zeta)$ is a holomorphic disk on S, there must exist a smooth map $z^0(\zeta)$ such that $z_{\mathrm{A}}(\zeta) = h_{\mathrm{A}}[z^0(\zeta), z_3(\zeta)]$, $(\mathrm{A} = 1, 2)$. Derivation with respect to $\bar{\zeta}$ gives that $z^0(\zeta)$ must be holomorphic, and since $|z^0(\zeta)| = 1$, constant. Therefore S does not contain other 1-dimensional complex submanifold than the lines of $\mathscr{F}$. In particular S is not a 2-cylinder. If S were a 1-cylinder $\phi(\zeta_1, \zeta_2) = 0$, then the generating lines $\zeta_3 = \text{var.}$ must be the family $\mathscr{F}$. Let $\zeta_j = f_j(z)$ be the change of coordinates, $z_j = g_j(\zeta)$ its inverse. By the previous argument $\partial g_3 / \partial \zeta_3$ is different from zero, so the equation $g_3(\zeta) = 0$ has an holomorphic solution $\zeta_3 = H(\zeta_1, \zeta_2)$. Consider the holomorphic function $\tilde{G}(z) = z_1 - g_1\{f_1(z), f_2(z), H[f_1(z), f_2(z)]\}$. There is $\tilde{G}[h_1(z^0, z_3), h_2(z^0, z_3), z_3] = z_1 - z_1^0$, for $|z^0| = 1$ near $(1, 0)$. Moreover $\tilde{G}(z_1, z_2, 0) = 0$. Hence $\tilde{G}(z) = z_3 G(z)$, and G is holomorphic near $(1, 0, 0)$. Then $G(z_1, z_2, 0) = \lambda(z_1)$ in a neighbourhood of $(1, 0)$ in $\mathbf{C}^2$ would be a continuation of λ past $z_1 = 1$. We still must prove that S satisfies (2) and (3). We will prove that S is the limit (uniform with derivatives) of 1-cylinders, indipendently of Theorem 1 (ii), since cylinders satisfy (3), S will also satisfy by continuity. Consider the hypersurface S_ε defined like S but with $|z^0| = 1 - \varepsilon$. For $\varepsilon > 0$ S_ε is a cylinder, with the coordinates ζ given by $z_A = h_A(\zeta_1, \zeta_2, z_3)$, $(A = 1, 2)$, $z_3 = \zeta_3$. S_ε are, by the way, the same cylinders of Theorem 1 (ii). Since S_ε has strictly pseudoconvex section it satisfies (3) and $S_\varepsilon \to S$. Finally S is pseudoconvex as C^2-limit of pseudoconvex hypersurfaces, so the lines $\mathscr{F}$ have, as complex tangent space the Levi kernel of S, hence S satisfies also (2) with $k = 2$.

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