General operators binding variables in the interpreted modal calculus $\mathcal{MC}''$


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Logica matematica. — General operators binding variables in the interpreted modal calculus $\mathcal{M}^G$. Nota di Aldo Bressan e Alberto Zanardo (*), presentata (**), dal Corrisp. A. Bressan.

Riassunto. — Si considera il calcolo modale interpretato MC$, che è basato su un sistema di tipi con infiniti livelli, contiene descrizioni, ed è dotato di una semantica di tipo generale — v. [2], o [3], o [4], o [5]. In modo semplice e naturale si introducono in MC$ operatori vincolanti variabili, di tipo generale. Per teorie basate sul calcolo logico risultante $\mathcal{M}^G$ vale un teorema di completezza, che si dimostra in modo immediato sulla base dell’estensione del teorema parziale di completezza stabilito in [11], fatta in [12].

1. INTRODUCTION

In extensional logic variable binding term operators, vbtos, or formula-term operators—i.e. expressions of the form $(\Omega, y_1, \cdots, y_n)$ that, if applied to a wff (well formed formula) generate a term—are studied from the semantical point of view in e.g. [7], and from the syntactical one in e.g. [8] and [9],

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where also completeness theorems are proved. In [9], where a survey on the subject is also presented, it is said——on p. 158—that future papers are planned to treat the vbtos in a modal logic, by means of a modified kind of Kripke models.

Here we consider general variable binding operators that apply to wffs or terms and generate wffs or terms; and we introduce them in a very simple way within the frame of the general modal calculus MC" whose underlying language ML" is based on a type system with infinitely many levels. There is no need of changing the semantics for ML". We use as operator signs functors and attributes already existing in ML". We simply add a new rule to form operator expressions and two natural axioms. Thus the language ML" and the calculus MCV" are obtained.

The completeness theorem for the 1st order fragment of MCV", deprived of the iota operator 1 for descriptions, is proved in [11]. This result is extended to the full calculus MC" in [12]. Here this extension is easily carried over to MCV". Incidentally the fact that this extension can deal with descriptions is essential. Indeed, from the present paper it appears that general operators can be reduced to descriptions; furthermore that the use of types simplifies the introduction of the above operators (0).

2. The modal language ML" and the modal calculus MCV" based on it

The language ML" (v ∈ Z+, the set of positive integers) is based on a type system τ" which is the smallest set such that {1, · · · , v} ⊆ τ" and ⟨t1, · · · , tn, t0⟩ ∈ τ" whenever t1, · · · , tn ∈ τ" and t0 ∈ τ0 := τ" ∪ {0}.

For every t ∈ τ", the constants ctn and the variables vtn (n ∈ Z+) are primitive symbols of ML" in addition to the usual logical symbols =, ∧, ∨, ◊, ♠, comma, and left and right parentheses. For t ∈ τ" the set ηt of the well formed expressions (wfe) of type t of ML" is defined recursively by formation rules (f1) to (fn) below, where t1, t2, · · · , tn run over τ" and t0 [n] runs over τ" [Z+].

\[(f1)\quad vtn, ctn ∈ ηt \land Δ1, Δ2 ∈ ηt ⇒ Δ1 = Δ2 ∈ η0;\]
\[(f2)\quad Δ1 ∈ ηi, (i = 1, · · · , n) \land Δ ∈ ηt1, · · · , tn, t0 ⇒ Δ (Δ1, · · · , Δn) ∈ η0;\]
\[(f3)\quad p, q ∈ η0 ⇒ (¬ p), (p ∧ q), ((\forall vtn)p), \text{and } (◊ p) ∈ η0;\]
\[(f4)\quad p ∈ η0 ⇒ (vtn)p ∈ ηt.\]

(1) General operators are treated within an extensional first order theory in [6]. The completeness theorem for them proved there has an essential role in [1]. The present treatment of operators in MCV" is much simpler.
The elements of $\mathcal{E}_0$ are called well formed formulas, and those of $\mathcal{E}_t (r \in \{1, \ldots, v\})$ are called individual terms. Furthermore, the elements of $\mathcal{E}_{(t_1, \ldots, t_n, 0)}$ are said to be relation [function] terms if $t = 0 [t_0 \in \tau^r]$.

The symbols $\forall, \exists, (\exists v_{m})$, $\diamond$, and other metalinguistic abbreviations are understood to be introduced in the usual way. In particular $(\exists t, x) p$ will stand for $(\exists v) (p \wedge (y) (p [x/y] \Rightarrow x = y))$, where $y$ is the first variable, of the same type of $x$, not free in $p$; and $\lambda$-expressions are introduced by the definitions

\begin{equation}
(\lambda x_1, \cdots, x_n) p = d (\forall x_1, \cdots, x_n) (F (x_1, \cdots, x_n) = p) \\
(\lambda x_1, \cdots, x_n) A = d (\forall x_1, \cdots, x_n) f (x_1, \cdots, x_n) = A
\end{equation}

where $x_1$ to $x_n$ are $n$ (distinct) variables in $\mathcal{E}_{t_1}$ to $\mathcal{E}_{t_n}$ respectively, $p [A]$ is a wff [a term of type $t_0$], and $F [f]$ is the first variable of type $(t_1, \ldots, t_n, 0) [(t_1, \cdots, t_n, t_0)]$ not free in $p [A]$. Furthermore, every expression used in the sequel is assumed to be well formed, which makes several explanations unnecessary.

A formula $p$ will be said to be modally closed if it is constructed from wffs $\Box p_1, \cdots, \Box p_7$ by means of $\neg, \wedge, (v_{m})$, and $\Box$.

The basic axiom schemes for MCV, i.e. those which determine the minimal version of this calculus, are A2.1-17 below. In them $p$ and $q$ denote wffs, $A$ denotes a term, and $x, y, z, x_1, \cdots, x_n, F, G, f$, and $g$ denote variables of suitable types.

A2.1-6 The axioms of the predicate calculus;

A2.7, 8 $\Box (p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q) ; \Box p \Rightarrow p$;
A2.9 $p \Rightarrow \Box p$, where $p$ is modally closed;
A2.10-12 $x := x ; x := y \wedge y = z \Rightarrow x := z ; \Box x := y \Rightarrow \Box [z/x] = \Box [z/y]$;
A2.13 $F = G \equiv (\forall x_1, \cdots, x_n) (F (x_1, \cdots, x_n) \equiv G (x_1, \cdots, x_n))$;
A2.14 $f = g \equiv (\forall x_1, \cdots, x_n) f (x_1, \cdots, x_n) = g (x_1, \cdots, x_n)$;
A2.15 $(\exists F) (\forall x_1, \cdots, x_n) (F (x_1, \cdots, x_n) = p)$;
A2.16 $(\exists f) (\forall x_1, \cdots, x_n) f (x_1, \cdots, x_n) = A$;
A2.17 (a) $(\exists v_{m}) p \wedge p [v_{m}/y] \Rightarrow y = (v_{m}) p$
(b) $\sim (\exists v_{m}) p \Rightarrow (v_{m}) p = (v_{m}) (v_{m} \neq v_{m})$.

Our deduction rules are the Modus Ponens and the generalization and necessitation rules restricted to axioms: if $\mathcal{A}$ is an axiom and $x_1$ to $x_n$ are variables, then $\Box (\forall x_1, \cdots, x_n) \mathcal{A}$ is a direct consequence of $\mathcal{A}$.
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For every choice of the υ + 1 sets D_1, ..., D_υ and Γ, we say that the set S = {2\mathcal{F}_t : t ∈ \tau'} is a QI-structure in case the following conditions (3.1–3) hold:

\begin{align}
&2\mathcal{F}_0 \subseteq \mathcal{P}(\Gamma) ; \quad 2\mathcal{F}_r \subseteq (\Gamma \rightarrow D_r) ; \\
&\quad 2\mathcal{F}(t_1, ..., t_ν, t_0) \subseteq ((\Pi_1^n 2\mathcal{F}_i) \rightarrow 2\mathcal{F}_0)
\end{align}

where α ∈ \mathcal{P}(β) ⇔ α ⊆ β and \Pi_1^n α_i = α_1 × ⋯ × α_n. If a^ν is a function of domain \tau^ν, such that (a^i = a^ν) t ∈ 2\mathcal{F}_i (for all t ∈ \tau'), then we say that the ordered pair (\mathcal{S}, a^ν) is a QI-system. We shall call a^ν the non-existing object of type t, looking forward to identify it with the designatum of every description (in \mathcal{E}_i) which does not fulfill its exact uniqueness condition—cf. rule (d_4) below. The elements of 2\mathcal{F}_i are called quasi-intensions (QI's) of type t.

An ML'-interpretation is an ordered triple \mathfrak{I} = (\mathcal{S}, a^ν, 2\mathcal{F}_i) in which \mathcal{S} is a valuation of the constants of ML^ν in \mathcal{S}, that is, a function assigning each c in a QI \mathcal{F}(c)n in 2\mathcal{F}_i. If in (3.1–3) the relation \subseteq holds as an equality, then \mathfrak{I} is said to be standard.

Definition 3.1. Let \mathfrak{I} be an ML'-interpretation and let ξ, ζ ∈ 2\mathcal{F}_i (t ∈ \tau'). Then ξ and ζ are said to be equivalent in the case γ ∈ Γ—briefly ξ =_ξ ζ—if one of the following conditions holds:

\begin{align}
(i) \quad [\text{[ii]}] \quad t = 0 [t ∈ \{1, ..., ν\}] \text{ and } ξ ∩ \{γ\} = ζ ∩ \{γ\} [ξ(γ) = ζ(γ)] ; \\
(iii) \quad t = (t_1, ..., t_ν, t_0) \text{ and, for all } α ∈ \Pi_1^n 2\mathcal{F}_i, ξ(α) = _ξ ζ(α).
\end{align}

Let now \mathfrak{I} be any ML'-interpretation. The set of all valuations of the variables of ML^ν in \mathfrak{I} will be denoted by Val_3. The following rules (d_1) to (d_8) define the designatum des_3^ν(\Delta) in \mathfrak{I} of the wfe \Delta, in correspondence with the valuation v ∈ Val_3. In these rules we assume des_3^ν(\Delta') = \Delta' for all subexpressions \Delta' of Δ, and x ∈ \mathcal{E}_i (t ∈ \tau').

\begin{align}
(d_1) \quad \text{des}_3^ν (v_m) = ν^ν (v_m), \text{des}_3^ν (c_m) = \mathcal{F}(c_m), (t ∈ \tau^ν, n ∈ \mathbb{Z}^+) ; \\
(d_2) \quad \text{des}_3^ν (Δ_1 = Δ_2) = \{γ ∈ Γ : \tilde{Δ}_1 =_γ \tilde{Δ}_2\}, (Δ_1, Δ_2 ∈ \mathcal{E}_i) ; \\
(d_3) \quad \text{des}_3^ν (Δ_0 (Δ_1, ..., Δ_ν)) = \tilde{Δ}_0 (\tilde{Δ}_1, ..., \tilde{Δ}_ν) ; \\
(d_4) \quad \text{des}_3^ν (\sim p) = Γ − \tilde{p} ; \text{des}_3^ν (p ∧ q) = \tilde{p} ∩ \tilde{q} ; \\
(d_5) \quad \text{des}_3^ν (v^ν(\Delta) p) = \{γ ∈ Γ : \text{ for all } ξ ∈ 2\mathcal{F}_i, γ ∈ \text{des}_3^ν (p) \text{ if } ν^ν / ν^ν(ξ)\};
\end{align}

(2) In e.g. [3] to [5] and [12], (t_1, ..., t_ν, t_0) is denoted by (t_1, ..., t_ν, t_0) for t_0 = 0 and by (t_1, ..., t_ν : t_0) for t_0 ∈ \tau'. Furthermore, 2\mathcal{F}(t_1, ..., t_ν) is defined to be a certain set isomorphic with the right hand side of (3.3).
A. BRESSAN e A. ZANARDO, General operators binding variables, etc.

\[ \text{(d)} \quad \text{des}_{3^\mathcal{F}}(\square p) = \Gamma [\varnothing] \quad \text{if} \quad \bar{p} = \bar{\Gamma} [\bar{p} \not\in \Gamma] ; \]

\[ \text{(d)} \quad \text{des}_{3^\mathcal{F}}((1 \cdot x) p) = \text{the only QI } \zeta \text{ such that:} \]

\( (a) \gamma \in \text{des}_{3^\mathcal{F}}((\exists x) p) \) and \( \gamma \in \text{des}_{3^\mathcal{F}'}(p) \) for \( \forall' = \forall' (\bar{\gamma}) \Rightarrow \zeta = \gamma \alpha' \).

The exact uniqueness of the QI \( \zeta \) that fulfills (a) and (b) is proved in [2] (N 11); let us remark however that \( \zeta \) (as well as other designata) may fail to be in \( \bigcup_{t \in \mathcal{T}} \mathcal{A}_t \). In any case, this unsatisfactory situation does not happen when general ML'-interpretations are dealt with—cf. def. 3.3 below—and we shall consider only these interpretations.

As usual we say that a wff \( p \) is true in \( \mathcal{S} \) if \( \text{des}_{3^\mathcal{F}}(p) = \Gamma \) for all \( \forall' \in \text{Val}_3 \).

The language ML' has been constructed with a view to use it endowed with standard ML'-interpretations (which are the most natural among the above interpretations). However, with respect to them, the completeness theorem obviously fails to hold, because the analogous fact occurs for extensional theories based on a type system—cf. [10] (3).

However, a wider class of ML'-interpretations—the so-called general interpretations (cf. [10] for the extensional case)—can be defined, which turns out to be sound for the completeness of MC'.

**Definition 3.2.** The QI \( \zeta \) of type \( \langle t_1, \cdots, t_n, t_0 \rangle \) is said to be definable in the ML'-interpretation \( \mathcal{S} \) if there exist (i) a \( \forall' \in \text{Val}_3 \), (ii) a finite set \( X = \{x_1, \cdots, x_n\} \) of variables of type \( t_1, \cdots, t_n \), respectively, and (iii) a wff \( \Delta \in \mathcal{E}_t \), such that

\[ \zeta = d(\Delta, X, \mathcal{S}, \forall') = \{\langle \zeta_1, \cdots, \zeta_n \rangle, \text{des}_{3^\mathcal{F}'}(\Delta) : \]

\[ \exists \eta \in \mathcal{F}_t (i = 1, \cdots, n) \quad \text{and} \quad \forall' = \forall' \left( x_1, \cdots, x_n \right) \left( \zeta_{i_1}, \cdots, \zeta_{i_m} \right) \}

**Definition 3.3.** The ML'-interpretation \( \mathcal{S} \) is said to be general if, for all \( t \in \mathcal{T}' \), every QI of type \( t \) definable in \( \mathcal{S} \) belongs to \( \mathcal{A}_t \).

Incidentally, if \( \mathcal{S} \) is a general ML'-interpretation and \( \forall' \in \text{Val}_3 \), then the equality: \( \text{des}_{3^\mathcal{F}'}((\lambda x_1, \cdots, x_n \Delta) = d(\Delta, \{x_1, \cdots, x_n\}, \mathcal{S}, \forall') \) holds for every wff \( \Delta \) and every set \( \{x_1, \cdots, x_n\} \) of variables (4). This shows that even

(3) A theory \( \mathcal{E}_N \) of natural numbers can be developed in MC'—c. [2] NN 27, 45—, and hence, by the Gödel's incompleteness theorem, there are infinitely many ways of choosing a formula \( p \), true in a given standard model \( \mathcal{M} \) and unprovable in MC'. If \( \mathcal{M} \) is any other model, then the sets of the (intensional) designata in \( \mathcal{M} \) and \( \mathcal{M}' \), of the terms (or constants) of \( \mathcal{E}_N \) are isomorphic in a suitable sense. Therefore \( p \) is true also in \( \mathcal{M}' \).

(4) The proof of this statement is similar to that of Theor. 6.1. in [12]; and it is based on the fact that \( (3t, t) (\forall x_1, \cdots, x_n) \square (F(x_1, \cdots, x_n) = p) \) and \( (3t, t) (\forall x_1, \cdots, x_n) \)

\[ \square f(x_1, \cdots, x_n) = \Delta \] are theorems of MC' for every wff \( p \) and every term \( \Delta \).
if $\mathcal{I}$ is a non-standard general ML$^\nu$-interpretation, $\lambda$-expressions have in it designata of a rather usual kind; in particular they cannot be the "non-existing object" as, a priori, could be surmised on the basis of (2.1).

The following theorem is proved (by a Henkin's method) in [12].

**Theorem 3.1.** (Completeness of MC$^\nu$). Every wff $p$ of ML$^\nu$ is provable in MC$^\nu$ [in a theory $\mathcal{E}$ based on MC$^\nu$ an having proper axioms] iff it is true in every general ML$^\nu$-interpretation [general model of $\mathcal{E}$].

4. **General operators in ML$^\nu$**

Let us assume that $t_1, \ldots, t_n \in \tau'$ and $\delta, \varepsilon \in \tau$, with either $n > 0$ or $n = 0 = \delta$. Intuitively an operator of type $w = (t_1, \ldots, t_n; \delta, \varepsilon)$ is an expression $\Omega$ such that, if $\Delta \in \mathcal{E}_0$ and $y_1$ to $y_n$ are $n$ variables of the respective types $t_1$ to $t_n$, then $(\Omega y_1, \ldots, y_n) \Delta \in \mathcal{E}$.

It is natural and useful to identify such operators with the elements of $\mathcal{E}_w$, under the definition

$$(4.1) \quad w = (t_1, \ldots, t_n; \delta, \varepsilon) = \begin{cases} \langle \langle t_1, \ldots, t_n, \delta \rangle, \varepsilon \rangle & (n > 0), \\ \langle 1, \delta \rangle, \varepsilon \rangle & (n = 0 = \delta). \end{cases}$$

Thus, first, we add $(f_1)$ to $(f_2)$ (N 2) with the formation rule $(f_3)$.

If (i) $\Omega \in \mathcal{E}(t_1, \ldots, t_n; \delta, \varepsilon)$, (ii) $y_1$ to $y_n$ are $n$ variables in $\mathcal{E}_t$ to $\mathcal{E}_t$, respectively, and (iii) $\Delta \in \mathcal{E}_0$, then $(\Omega y_1, \ldots, y_n) \Delta \in \mathcal{E}_n (n > 0, or \delta = 0 = n)$;

and, secondly, we add A2.1–17 with

| A4.1 | $(\Omega y_1, \ldots, y_n) \Delta = \Omega ((\lambda y_1, \ldots, y_n) \Delta)$ for $n > 0$ and $\varepsilon = 0$ |
| A4.2 | $\Omega ((\lambda y_1, \ldots, y_n) \Delta) = \Omega (\lambda y_1, \ldots, y_n) \Delta$ for $n > 0 = \delta$ and $\varepsilon = 0$ |
| A4.3 | $(\Omega) \Delta = \Omega (\lambda y) \Delta$ for $n = 0 = \delta$ and $\varepsilon = 0$ |
| A4.4 | $\varepsilon \in \tau'$ |

where (i) to (iii) in $(f_3)$ hold and (iv) $y$ is the first variable of type 1 that fails to occur in $\Delta$. Let us call $\mathcal{ML}^\nu$ [$\mathcal{MC}^\nu$] what ML$^\nu$ [MC$^\nu$] becomes by the addition of $(f_3)$ [AA.4.1–4].

In order to construct the semantics for $\mathcal{ML}^\nu$, we assume that every [every general] ML$^\nu$-interpretation $\mathcal{I}$ is also a [a general] $\mathcal{ML}^\nu$-interpretation and that, for all $\mathcal{V} \in \text{Val}_3$,

$$(d_\nu) \quad \text{des}_{\mathcal{ML}^\nu} ((\Omega y_1, \ldots, y_n) \Delta) = \text{des}_{\mathcal{ML}^\nu} (\Omega ((\lambda y_1, \ldots, y_n) \Delta)) \quad (n > 0),$$

$$\text{des}_{\mathcal{ML}^\nu} ((\Omega) \Delta) = \text{des}_{\mathcal{ML}^\nu} (\Omega (\lambda y) \Delta) \quad \text{where (iv) holds} \quad (n = 0 = \delta).$$

In this way we can immediately conclude that the analogue for $\mathcal{ML}^\nu$ and $\mathcal{MC}^\nu$ of Theorem 3.1, the completeness theorem, holds.
The case \( n = 0 = \theta \) is interesting by the following reasons. If \( F[f] \) is a variable not free in the wff \( p \), of the type \( (; 0, 0) [([0, t_R], \text{where } t_R \text{ is the type for real numbers}] \), then a suitable \( \forall \) exists, for which \((a)\) \((b)\) below holds.

\[(a) \ \text{des}_\forall^\forall ((F)p) \text{ equals any preassigned of the designata des}^\forall_\forall (\Box p) \text{ and des}^\forall_\forall (\Diamond p).\]

\[(b) \ \text{des}^\forall_\forall(q) = \Gamma, \text{ where } q \text{ is } (f)p = 1 \land p \lor (f)p = 0 \land \sim p.\]

In case \((b)\) \(\text{des}^\forall_\forall ((f)p)\) is the indicator of \( p \) (in \( \exists \) and \( \forall \)), i.e., the characteristic function of \( \text{des}^\forall_\forall(p)\).

References


