Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Massimo Lorenzani

Optimal stopping for Markov Processes

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **70** (1981), n.2, p. 101–108.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1981_8_70_2_101_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1981.

Calcolo delle probabilità. — *Optimal stopping for Markov Processes.* Nota di MASSIMO LORENZANI, presentata (*) dal Socio E. MARTINELLI.

RIASSUNTO. — In questa nota presentiamo dei nuovi risultati sul problema di tempo d'arresto ottimale per processi di Markov con tempo discreto.

1. In this note we present some new results concerning the existence of an optimal stopping time for Markov processes with discrete time.

We shall follow the notations of [10] a) or b). Let $X = (X_n, \mathcal{F}_n, P_x)$, $n \in \mathbf{N}$, be a Markov process with state space $(E, \mathcal{B})$, where $\mathcal{B}$ is a σ-algebra, and transition probability $p(x, \Gamma)$. We denote by L the space of measurable functions $f : E \longrightarrow [-\infty, +\infty)$ such that $M_x\{|f(X_n)|\} < +\infty$, for every $n \in \mathbf{N}$, and by $\mathbf{B}$ the Banach space of bounded measurable functions on E (of course, $\mathbf{B} \subseteq L$). Since $M_x\{f(X_1)\} = \int_E f(y)\, p(x, dy)$, the right hand side defines a linear operator on L or $\mathbf{B}$ which will be denoted by T. If $f(x), g(x) \in L$, we introduce the *value function* $\tilde{v}(x)$ as

$$\tilde{v}(x) = \sup_{\tau \in \mathfrak{M}_{(f,g)}} M_x \left\{ \sum_{k=0}^{\tau-1} f(X_k) + g(X_\tau) \right\}$$

where $\mathfrak{M}_{(f,g)}$ is the class of stopping times such that

$$M_x \left\{ \sum_{k=0}^{\tau-1} f^-(X_k) + g^-(X_\tau) \right\} < +\infty.$$

We also introduce the function

$$v(x) = \sup_{\tau \in \mathfrak{M}^b_{(f,g)}} M_x \left\{ \sum_{k=0}^{\tau-1} f(X_k) + g(X_\tau) \right\},$$

with $\mathfrak{M}^b_{(f,g)} = \{\tau \in \mathfrak{M}_{(f,g)} / \tau \text{ is bounded}\}$, which is, moreover, the limit of the functions $g_n(x) = \max\{g_{n-1}(x), f(x) + \mathrm{T}g_{n-1}(x)\} = \max\{g(x), f(x) + \mathrm{T}g_{n-1}(x)\}$, $g_0(x) = g(x)$ (see [10] b), pag. 28). Furthermore, $v(x)$ satisfies the equation

$$v(x) = \max\{g(x), f(x) + Tv(x)\},$$

(*) Nella seduta del 14 febbraio 1981.

since it is the minimal solution of the system of inequalities

$$(S) \qquad \begin{cases} u(x) \geq g(x) \\ u(x) - \mathrm{T}u(x) \geq f(x) \end{cases}$$

(we observe that also $\tilde{v}(x)$ is a solution of (S)).

2. General results

The main problem (or Snell's problem, see [9]) related to the function $\tilde{v}(x)$ is the following: *find a stopping time* $\tilde{\tau} \in \mathfrak{M}_{(f,g)}$ *such that*

$$\tilde{v}(x) = M_x \left\{ \sum_{k=0}^{\tau-1} f(\mathrm{X}_k) + g(\mathrm{X}_{\tilde{\tau}}) \right\};$$

i.e., an optimal stopping time.

All the results we shall obtain on the existence of an optimal stopping time are based on the following.

Theorem 1. *Let* $f(x), g(x) \in \mathrm{L}$ *be two functions such that:*

i) $f(x) = (\mathrm{T} - \mathrm{I})\, z(x) - w(x)$, *with* $z(x)$, $w(x) \in \mathrm{L}$, $w(x) \geq 0$ ($\mathrm{I} =$ *identity operator*)

ii) $\mathrm{M}_x \{\sup_n |(g+z)(\mathrm{X}_n)|\} < +\infty$.

Then,

1) $\tilde{v}(x) = v(x)$.

2) $\tau_\varepsilon = \min \{k \geq 0 / v(\mathrm{X}_k) \leq g(\mathrm{X}_k) + \varepsilon\}$ *is an* ε*-optimal stopping time; that is,*

$$v(x) \geq \mathrm{M}_x \left\{ \sum_{k=0}^{\tau_\varepsilon - 1} f(\mathrm{X}_k) + g(\mathrm{X}_{\tau_\varepsilon}) \right\} \geq v(x) - \varepsilon.$$

3) *If* $\mathrm{P}_x \{\tau_0 < +\infty\} = 1$, τ_0 *is optimal in* $\mathfrak{M}_{(f,g)}$.

Proof (sketch). The sequence of random variables

$$\Psi_n = z(\mathrm{X}_n) + \sum_{i=0}^{n-1} (\mathrm{I} - \mathrm{T})\, z(\mathrm{X}_i)$$

is an $\mathscr{F}_n$-martingale. Since we have the relation $M_x\{\Psi_0\} = M_x\{\Psi_\tau\}$ for every bounded stopping time τ (see [9], prop. IV–3–13), from *i)* it follows that

$$M_x \left\{ \sum_{k=0}^{\tau-1} f(\mathrm{X}_k) \right\} = M_x \left\{ z(\mathrm{X}_\tau) - \sum_{k=0}^{\tau-1} w(\mathrm{X}_k) \right\} - z(x).$$

The above relation shows that $v(x)$ can be also obtained starting from the functions $(g+z)(x)$ and $-w(x)$. Thus, theor. 1 follows from [10] a), teor. 16 or b), theor. 23.

Remark 1. If $g(x)$, $z(x) \in \mathbf{B}$, the condition *ii*) is, of course, satisfied; and, if also $w(x) \in \mathbf{B}$, $v(x) \in \mathbf{B}$.

From now on we shall deal with bounded functions.

A probability measure μ *on* $(E, \mathscr{B})$ *is said an invariant measure for the Markov process if*

$$\mu(\Gamma) = \int_E p(x, \Gamma)\, \mu(dx), \qquad \forall \Gamma \in \mathscr{B}.$$

Let us consider a Markov process with an invariant measure μ. If $f(x)$ is as in theor. 1, since $v(x) \in \mathbf{B}$, we observe that $f(x)$ satisfies necessarily the following relation

$$\int f(x)\, \mu(dx) \leq 0.$$

Indeed, since μ is an invariant measure, we have $\int_E z(x)\, \mu(dx) = \int_E Tz(x)\, \mu(dx)$ for every $z(x) \in \mathbf{B}$; and then it is enough to make use of the second inequality of (S). Of course, the above relation does not imply, in general, that the function $f(x)$ verifies the condition *i*) of theor. 1. However, the following proposition shows when this is possible.

PROPOSITION 1. *Let* $X = (X_n, \mathscr{F}_n, P_x)$, $n \in \mathbf{N}$, *be a Markov process with an invariant measure* μ. *If the linear operator* $T : \mathbf{B} \to \mathbf{B}$ *is such that* $\operatorname{codim}_{\mathbf{R}} R(T - I) = 1$ ($R(T - I)$ *is then a closed subspace of* $\mathbf{B}$), *the following conditions are equivalent*:

i) $f(x) = (T - I)\, z(x) - w(x)$, $w(x) \geq 0$

ii) $\int f(x)\, \mu(dx) \leq 0$, *for an* (*and then all*) *invariant measure* μ.

In [4] can be found examples of Markov processes for which $\operatorname{codim}_{\mathbf{R}}$ $R(T - I) = 1$.

3. MARKOV CHAINS

If we consider the class of Markov chains, we can get more precise results on Snell's problem. To this purpose, we consider two cases: 1) E finite, 2) E infinite.

1) Since every finite Markov chain has an invariant measure, we have the following

THEOREM 2. *If* E *is finite, the conditions*

i) $f(x) = (T - I)\, z(x) - w(x)$, $w(x) \geq 0$,

ii) $\langle f, \mu \rangle < 0$ ($\langle f, \mu \rangle = \sum_{x \in E} f(x)\, \mu(x)$) *for every invariant measure* μ, *are equivalent. Moreover, τ_0 is an optimal stopping time if and only if one of the above conditions is satisfied.*

2) If E is infinite, a Markov chain has not in general an invariant measure. Indeed, if E_p denotes the set of positive recurrent states, there exists an invariant measure μ if and only if $E_p \neq \varnothing$. Furthermore, such a measure is unique if and only if E_p is an irreducible class of states. Making use of the above properties, we obtain the following

THEOREM 3. *Let* $X = (X_n, \mathcal{F}_n, P_x)$, $n \in \mathbf{N}$, *be an irreducible Markov chain (i.e.* $E_p =$ *unique irreducible class). If* $E = E_p$ *and* $f(x) = (T - I)\, z(x) + - w(x)$, $w(x) \geq 0$ *and not identically zero,* τ_0 *is an optimal stopping time.*

4. Ergodic transformations

A very interesting class of Markov processes is given by the *dynamical systems*. We recall some definitions and properties from [2] and [11].

A *dynamical system* is a collection (E, μ, ϕ_n), $n \in \mathbf{Z}$, where E is a measurable space, μ a finite measure (we may assume that $\mu(E) = 1$) and ϕ_n a (discrete) one-parameter group of automorphisms of (E, μ) for which μ is invariant; i.e.,

$$\mu(\phi_n(A)) = \mu(A) \qquad \forall n \in \mathbf{Z}, \forall A \quad \text{measurable.}$$

– A dynamical system (E, μ, ϕ_n) is said *ergodic* if, for each $f(x) \in L^1(E, \mu)$, we have

$$\lim \frac{1}{n} \sum_{k=0}^{n-1} f(\phi_n(x)) = \int_E f(x)\, \mu(dx)\,, \qquad \text{a.e. in } E.$$

Let (E, μ, ϕ_n) be a dynamical system, we may define a group of linear operators $\{T^n\}$, $n \in \mathbf{Z}$, on $L^1(E, \mu)$ by means the relation:

$$T^n f(x) = f(\phi_n(x))\,,$$

(we put $T^0 = T$). Then, according to [5], theor. 2.1, a dynamical system can be viewed as a Markov process; therefore, we have the following.

THEOREM 4. *Let* (E, μ, ϕ_n) *be an ergodic dynamical system and* $f(x), g(x) \in \mathbf{B}$. *If* $f(x) = (T - I)\, z(x) - w(x)$, *with* $w(x) \geq 0$ *and* $w(x) > 0$ *an a set positive measure,* τ_0 *is an optimal stopping time.*

The results obtained show that only in the case of finite Markov chains we can give a necessary and sufficient condition for the solution of the Snell's problem. In the other cases, that condition (on the function $f(x)$) is only sufficient. Therefore, we can ask what happens if $f(x)$ is a little more general. To this purpose, we will study the optimal stopping problem on a classical example of ergodic dynamical system: *the rotation of the circle.*

Let $E = S^1 = \{e^{2\pi i x}, x \in \mathbf{R}\}$ be a circle, $\alpha \in (0, 1)$ an irrational number and ϕ^α the automorphism of S^1 defined by the relation

$$\phi^\alpha(e^{2\pi i x}) = e^{2\pi i (x+\alpha)},$$

$(\phi_n^\alpha(e^{2\pi i x}) = e^{2\pi i(x+n\alpha)}, n \in \mathbf{Z})$. Then, $(S^1, \mu, \phi_n^\alpha)$ is an ergodic dynamical system (see [11]), where μ is the Lebesgue measure. Since the system $(S^1, \mu, \phi_n^\alpha)$ is a deterministic Markov process we have

$$\tilde{v}(x) = v(x) = \sup_n M_x \left\{ \sum_{k=0}^n f(X_k) \right\} = \sup_n \left\{ \sum_{k=0}^n T^k f(x) \right\} =$$

$$= \sup_n \left\{ \sum_{k=0}^n f(\phi_k^\alpha(x)) \right\}, \qquad f \in L^1(S^1, \mu)$$

(we assume that $f(x) = g(x)$).

We shall consider three cases:

1) $\int_{S^1} f(x)\,\mu(dx) > 0$, 2) $\int_{S^1} f(x)\,\mu(dx) = 0$, 3) $\int_{S^1} f(x)\,\mu(dx) < 0$.

1) Since the dynamical system (S^1, μ, ϕ^α) is ergodic,

$$\lim \frac{1}{n+1} \sum_{k+0}^n f(\phi_k^\alpha(x)) > 0.$$

Therefore, $\sup_n \left\{ \sum_{k=0}^n f(\phi_k^\alpha(x) \right\} = \tilde{v}(x) = +\infty$ (1). Observe that a function with positive mean cannot verify *i*) of the theor. 1.

2) In this case, $f(x)$ satisfies *i*) of the theor. 1 if and only if $f(x) \in R(\mathrm{T} - \mathrm{I})$; in other words, if and only if the equation $Tz(x) + - z(x) = f(x)$ has a solution in $L^1(S^1, \mu)$.

Let us assume that $f(x) \in C^r(S^1)$, $r \geq 0$. Then, theor. 14.13 of [6] (see also [8], prop. 5.7.3, p. 193) shows that

$$\sup_n \left\{ \sum_{k=0}^n f(\phi_k^\alpha(x)) \right\} = +\infty$$

if and only if the equation $Tz(x) - z(x) = f(x)$ has not a solution in $C^0(S^1)$; moreover, if the above equation has not continuous solutions, cannot be solved also in $L^\infty(S^1, \mu)$ (see [8], prop. 4.2, p. 45). Therefore, we may ask when the equation has continuous solutions. It can be shown that the regularity of the solution depends on the choice of the irrational number α. Indeed, in [2], p. 224 and [8], prop. 8.2.1., p. 230 are given conditions on α (in the sense

(1) This fact is true for every ergodic dynamical system.

of the theory of diophantine approximation) which guarantee the existence of continuous solutions. But, there exists irrational numbers α (precisely, those *well approximable*) for which it is not hard to see that the equation has not solutions in $C^0(S^1)$ (see [2], p. 225). The following two results give an idea on the various possibilities of solutions and are very useful for our purpose.

In [1], Anosov has constructed an analytic function $f(x)$ and an irrational number α such that $\mathrm{T}z(x) - z(x) = f(x)$ has a *measurable* solution which is not in $L^1(S^1, \mu)$. In [7], Herman showed that, if $f(x) \in L^2(S^1, \mu)$ has a *lacunary Fourier expansion*, the equation has always a solution in $L^2(S^1, \mu)$. Then, even if $f(x)$ verifies the condition *i*) of the theor. 1, with $z(x) \notin C^0(S^1, \mu)$ (and then $z(x) \notin L^\infty(S^1, \mu)$), the value function $\tilde{v}(x) = +\infty$. Therefore, *i*) of the theor. 1 is the best condition under which the optimal stopping problem makes sense.

3) If the function $f(x)$ has negative mean, it is possible, as showed in 2), that it does not verify *i*) of the theor. 1. But the ergodicity property tell us that

$$\lim \frac{1}{n+1} \sum_{k=0}^{n} f(\phi_n^\alpha(x)) < 0 .$$

Then, $\sum_{k=0}^{n} f(\phi_k(x)) < 0$ a.e. for $n \gg 0$. If we choose $f(x)$ as in 2), the function $v(x)$ is finite a.e. But we are not able to prove if 3) is enough to guarantee that $v(x) \in \mathrm{L}^1(\mathrm{E}, \mu)$ or the existence of an optimal stopping time.

5. Optimal stopping with control

In the following we shall make use of the notations of [4]. Let us consider a Markov process with transition probability depending on a parameter; that is, we have a function $p(x, u, \Gamma) : E \times U \times \mathscr{B} \to \mathbf{R}$, where U is a set called *controls space*. A *control* is a sequence $V = (v_0, v_1, \cdots, v_n, \cdots)$ of measurable functions, where $v_0 : E \to U, v_1 : E \times E \to U, v_2 : E \times E \times E \to U, \cdots$. Moreover, a control is said *Markovian* if $v_n : E \to U$ for every $n \geq 0$. We denote by $\mathscr{V}$ the class of all the controls.

If $h(x)$ is a bounded and measurable function on E, for every $u \in \mathrm{U}$ we may define a linear operator on $\mathbf{B}$ by the relation

$$\mathrm{T}^u h(x) = \int_E h(y)\, p(x, u, \mathrm{d}y) .$$

Let $f(x, u) : E \times U \to \mathbf{R}, g(x) : E \to \mathbf{R}$ be two bounded measurable functions. Then, we may define the function

$$\tilde{v}(x) = \sup_{\tau \in \mathfrak{M}^{\mathscr{V}}, V \in \mathscr{V}} \mathrm{M}_x^V \left\{ \sum_{k=0}^{\tau-1} f(\mathrm{X}_k, v_k) + g(\mathrm{X}_\tau) \right\}$$

where $\mathfrak{M}^{\mathscr{V}}$ is the class of stopping times such that

$$M_x^V \left\{ \sum_{k=0}^{\tau-1} f^-(\mathrm{X}_k, v_k) + g^-(\mathrm{X}_\tau) \right\} < +\infty$$

for every $V \in \mathscr{V}$. Let $\{g_n(x)\}$, $n \in \mathbf{N}$, be the sequence of functions defined in the following way:

$$g_n(x) = \max \{g(x), \sup_u [f(x, u) + \mathrm{T}^u g_{n-1}(x)]\},$$

with $g_0(x) = g(x)$. Then, the function $v(x) = \lim g_n(x)$ satisfy the equantion

$$v(x) = \max \{g(x), \sup_u [f(x, u) + \mathrm{T}^u v(x)]\};$$

and, moreover,

$$v(x) = \sup_{\tau \in \mathfrak{M}^V \text{bound.}, V \in \mathscr{V}} M_x^V \left\{ \sum_{k=0}^{\tau-1} f(\mathrm{X}_k, v_k) + g(\mathrm{X}_\tau) \right\}.$$

A control $\hat{V} = (\hat{v}_0, \hat{v}_1, \cdots, \hat{v}_n, \cdots)$ is said *optimal* if

$$g_n(x) = g_n^{\hat{V}}(x) = \max \{g(x), f(x, \hat{v}_{n-1}(x)) + \mathrm{T}^{\hat{v}_n} g_{n-1}(x)\}$$

for every $n \geq 0$.

THEOREM 5. *Let $f(x, u)$, $g(x)$ be two bounded measurable functions and assume that there exists a Markovian optimal control $\hat{\mathrm{V}}$. If $f(x, u) = (\mathrm{T}^u - \mathrm{I}) z(x) - w(x, u)$, with $w(x, u) \geq 0$, then*

1) $\tilde{v}(x) = v(x)$.

2) $\tau_\varepsilon = \min \{k \geq 0 / v(\mathrm{X}_k) \geq g(\mathrm{X}_k) + \varepsilon\}$ *is an optimal stopping time in* $\mathfrak{M}^V$, *that is*

$$v(x) \geq \mathrm{M}_x^{\hat{\mathrm{V}}} \left\{ \sum_{k=0}^{\tau_\varepsilon - 1} f(\mathrm{X}_k, \hat{v}_k) + g(\mathrm{X}_{\tau_\varepsilon}) \right\} \geq v(x) - \varepsilon.$$

3) *If* $\mathrm{P}_x^{\hat{\mathrm{V}}} \{\tau_0 < +\infty\} = 1$, τ_0 *is an optimal stopping time in* $\mathfrak{M}^{\mathscr{V}}$.

The existence of a Markovian optimal control is proved in [4] under the following hypothesis:

1) E metric space, 2) U compact metric space, 3) $f(x, u)$, $g(x)$ upper semi-continuous functions, 4) $T^u h(x)$ continuous in x and u if $h(x)$ is uniformly continuous.

BIBLIOGRAPHY

[1] D. V. ANOSOV (1973) – *On an additive functional homology equation connected with an ergodic rotation of the circle*, « Trans. Math. U.S.S.R. Izv. », *74*.

[2] V. A. ARNOLD (1965) – *Small denominators I. Mappings of the circumference onto itself*, « Am. Math. Soc. Transl. », *46*.

[3] V. A. ARNOLD and A. AVEZ (1967) – *Problèmes Ergodique de la Mécanique classique*, Gauthier–Villars.

[4] A. BENSOUSSAN – *Quinze Exposés sur les processus de Markov avec temps discret*. Not published.

[5] E. B. DYNKIN (1965) – *Markov Processes*, vol. I, Springer-Verlag.

[6] W. H. GOTTSCHALK and G. A. HEDLUND (1965) – *Topological Dynamics*, « Am. Math. Soc. Coll. Publ. », *36*.

[7] M. R. HERMAN (1976) – L^2 *Regularity of measurable solutions of a finite difference equation of the circle*, « Math. Inst. Univ. of Warwick ».

[8] M. R. HERMAN (1979) – *Sur la conjugaison differentiable des difféomorphismes du cercle a des rotations*, « Publ. Math. I.H.E.S. », *49*.

[9] J. NEVEU (1975) – *Discrete-Parameter Martingales*, North–Holland.

[10] A. N. SHIRYAYEV a) (1973) – *Statistical Sequential Analysis*, « Transl. of Math. Monog., Am. Math. Soc. ». b) (1978) *Optimal Stopping rules*, Springer–Verlag.

[11] Y. G. SINAY (1976) – *Introduction to Ergodic theory*, « Math. Notes », Princ. Un. Press.