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Differential Parameters in the Normal Gravity Field

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Geodesia. — *Differential Parameters in the Normal Gravity Field.* Nota [(*)] del Socio ANTONIO MARUSSI.

RIASSUNTO. — Dopo svolte alcune considerazioni sulla dualità fra i triedri di base e reciproci in un sistema di coordinate generali nello spazio metrico, si danno le espressioni per gli operatori differenziali elementari applicati ai vettori dei due triedri, con applicazione al caso delle coordinate intrinseche nel campo normale di gravità.

1. Given a system of general coordinates x^i $(i = 1, 2, 3)$ in the three-dimensional metric space, at each point P two triads of reference vectors are defined:

(1.1) the base triad $\boldsymbol{v}_i = \dfrac{\partial \mathrm{P}}{\partial x^i}$

(1.2) the reciprocal triad $\boldsymbol{v}^i = \operatorname{grad} x^i.$

The two triads are connected by the following equations:

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = g_{ij} \quad ; \quad \boldsymbol{v}_i \cdot \boldsymbol{v}^j = \delta_i^j \quad ; \quad \boldsymbol{v}^i \cdot \boldsymbol{v}^j = g^{ij} \tag{1.3}$$ [(1)]

in which g_{ij} is the metric tensor, g^{ij} its reciprocal, and δ_i^j the Kronecker's symbol.

For an elementary *displacement vector* we have

$$\mathrm{dP} = \frac{\partial \mathrm{P}}{\partial x^i}\, \mathrm{d}x^i = \boldsymbol{v}_i\, \mathrm{d}x^i \tag{1.4}$$

and for a *gradient vector*

$$\operatorname{grad} \varphi = \frac{\partial \varphi}{\partial x^i} \operatorname{grad} x^i = \boldsymbol{v}^i \frac{\partial \varphi}{\partial x^i}\,. \tag{1.5}$$

The natural function of the base vectors is therefore to describe *displacement vectors* or *contravariant tensors*; and the natural function of the reciprocal vectors is to describe *gradient vectors* or *covariant tensors*.

The distinction between the two kinds of vectors and tensors is essential in Affine Geometry; but it still preserves a definite character in Metric Geometry.

(*) Presentata nella seduta del 6 dicembre 1980.

(1) Summation over duplicated indices throughout the paper.

2. As has been shown in a paper dedicated to prof. Tauno Kukkamaki on the occasion of his 70th birthday, the two reference triads are indeed not equivalent since:

i) the base vectors $\boldsymbol{v}_i$ are not in general gradient vectors, as the reciprocal vectors are;

ii) the reciprocal vectors $\boldsymbol{v}^i$ are not in general the vectors of the principal triad of any triple family of surfaces, as the base vectors are.

Therefore, the Pfaffian form

$$\mathrm{d}x_i = g_{ij}\,\mathrm{d}x^j = \boldsymbol{v}_i \cdot \mathrm{dP} \tag{2.1}$$

is in general not a perfect differential and as a consequence the parallelogrammoid defined by two displacements

$$\mathrm{dP} = \boldsymbol{v}^i\,\mathrm{d}x_i \quad , \quad \delta\mathrm{P} = \boldsymbol{v}^i\,\delta x_i \tag{2.2}$$

will show a misclosure $\Delta\mathrm{P}$ given by

$$\Delta\mathrm{P} = (\delta\mathrm{d} - \mathrm{d}\delta)\,\mathrm{P} = \Omega^{ij\cdot}_{\cdot\cdot h}\,\boldsymbol{v}^h\,\mathrm{d}x_i\,\delta x_j \tag{2.3}$$

and in general $\Omega^{ij\cdot}_{\cdot\cdot h} \neq 0$.

Contrary to the coordinates x^i, the coordinates x_i defined by (2.1) are therefore non-holonomic.

The non-holonomity of the coordinates x_i is highlighted if we consider the *coefficients of connection* between neighboring triads as defined by

$$\mathrm{d}\boldsymbol{v}_i = \left\{ {h \atop ij} \right\} \boldsymbol{v}_h\,\mathrm{d}x^j \quad ; \quad \mathrm{d}\boldsymbol{v}^i = \left\{ {ij \atop h} \right\} \boldsymbol{v}^h\,\mathrm{d}x_j\,; \tag{2.4}$$

for which

$$\left\{ {h \atop ij} \right\} - \left\{ {h \atop ji} \right\} \equiv 0 \tag{2.5}$$

$$\left\{ {ij \atop h} \right\} - \left\{ {ji \atop h} \right\} = 2\,\Omega^{ij\cdot}_{\cdot\cdot h}\,. \tag{2.6}$$

As already remarked, $\Omega^{ij\cdot}_{\cdot\cdot h}$ is in general different from, zero and is called the *object of anholonomity* relevant to the coordinates x_i.

The symbols $\left\{ {h \atop ij} \right\}$ are the *symbols of Christoffel* of the second kind; and we obviously have

$$\left\{ {ij \atop h} \right\} = -\,g^{rj} \left\{ {i \atop hr} \right\}. \tag{2.7}$$

Indicating by (ij, h) the *symbols of Christoffel* of the first kind

$$(ij, h) = g_{hk} \left\{ \begin{matrix} k \\ i\, j \end{matrix} \right\} \tag{2.8}$$

and putting

$$\Omega_{ij,h} = g_{ir}\, g_{js}\, \Omega^{rs}_{..h} \tag{2.9}$$

whe also have

$$2\, \Omega_{ij,h} = (hi, j) - (hj, i)\,. \tag{2.10}$$

3. We shall in the following consider the differential operators curl and divergence.

As for the operator curl we have

$$\text{curl}\ \boldsymbol{v}^i = \text{curl grad}\ x^i \equiv 0 \tag{3.1}$$

and from the definition of the operator

$$\text{curl}\ \boldsymbol{v}_h \cdot \boldsymbol{v}_i \times \boldsymbol{v}_j = (hi, j) - (hj, i) = 2\, \Omega_{ij,h} = \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{hi}}{\partial x^j}\,. \tag{3.2}$$

We may also write

$$\text{curl}\ \boldsymbol{v}_h = \frac{1}{\sqrt{G}} \begin{vmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \\ \dfrac{\partial}{\partial x^1} & \dfrac{\partial}{\partial x^2} & \dfrac{\partial}{\partial x^3} \\ g_{h1} & g_{h2} & g_{h3} \end{vmatrix} \tag{3.3}$$

in which

$$G = |\, g_{rs} \,|\,. \tag{3.4}$$

We therefore have for any vector $\boldsymbol{u} = u_i\, \boldsymbol{v}^i$

$$\text{curl}\ \boldsymbol{u} = \frac{\partial u_i}{\partial x^j}\, \boldsymbol{v}^i \times \boldsymbol{v}^j\,. \tag{3.5}$$

4. Considering the operator divergence, we have

$$\text{div}\ \boldsymbol{v}^i = \left\{ \begin{matrix} ik \\ h \end{matrix} \right\} = \Delta_2\, x^i = -\, g^{hk} \left\{ \begin{matrix} i \\ h\,k \end{matrix} \right\} = \tau^i \tag{4.1}$$

and therefore

$$\text{div}\ \boldsymbol{u} = g^{ij} \frac{\partial u_i}{\partial x^j} + u_i\, \tau^i\,. \tag{4.2}$$

Conversely,

$$\text{div } \boldsymbol{v}_i = \left\{ {h \atop ih} \right\} = \frac{\partial}{\partial x^i} \lg \sqrt{G} = S_{/i} \tag{4.3}$$

and therefore

$$\text{div } \boldsymbol{u} = g^{ij} \frac{\partial u_i}{\partial x^j} + u^i S_{/i} \tag{4.4}$$

in which

$$S = \lg \sqrt{G} \,. \tag{4.5}$$

Consequently, the divergence of the base vectors as well as the divergence of the reciprocal vectors, are the components of two vectors respectively, both characteristic of the assigned coordinate system:

– the divergence of the base vectors, the covariant components of the logarithmic gradient of the volume spanned by the base vectors themselves:

$$\text{grad } S = \boldsymbol{v}^i \text{ div } \boldsymbol{v}_i = S_{/i} \, \boldsymbol{v}^i = \left\{ {h \atop ih} \right\} \boldsymbol{v}^i ; \tag{4.6}$$

– the divergence of the reciprocal vectors, the contravariant components of the displacement vector $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \boldsymbol{v}_i \text{ div } \boldsymbol{v}^i = \boldsymbol{v}_i \, \Delta_2 \, x^i = \left\{ {ih \atop h} \right\} \boldsymbol{v}_i \,. \tag{4.7}$$

It should also be noted that

$$2 \, \Omega^{ik}_{\cdot\cdot h} = \tau^i + S^{/i} \quad ; \quad S^{/i} = g^{ir} S_{/r} \,. \tag{4.8}$$

5. In the following an application is made to the case of the intrinsic coordinates $x^1 = \varphi$ (geodetic latitude), $x^2 = \lambda$ (geodetic longitude), $x^3 = U$ (normal potential) of the normal gravity field based on the standard biaxial ellipsoid for which the metric tensor is defined by

$$ds^2 = \rho^2 \, d^2\varphi + \nu^2 \cos^2 \varphi \, d\lambda^2 + \frac{1 + f^2}{\gamma^2} \, dU^2 + \frac{2 \, \rho f}{\gamma} \, d\varphi \, dU \,. \tag{5.1}$$

Here ρ and ν are the radii of curvature along the meridian and the prime vertical respectively, γ the normal gravity, and

$$f = \frac{\partial \lg \gamma}{\partial \varphi} \,. \tag{5.2}$$

It is hardly necessary to note that the same formulae are rigorously valid for any field endowed with rotational symmetry.

Indicating by $(\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$ the local orthonormal triad in which $\boldsymbol{i}_1$ is directed to the North, $\boldsymbol{i}_2$ to the East, and $\boldsymbol{i}_3$ to the Zenith, we have

$$\text{(5.3)} \qquad \text{curl}\, \boldsymbol{i}_1 = \frac{\boldsymbol{i}_2}{\rho} \quad ; \quad \text{curl}\, \boldsymbol{i}_2 = \frac{\boldsymbol{i}_1}{\nu} - \frac{\boldsymbol{i}_3 \operatorname{tg} \varphi}{\nu} \quad ; \quad \text{curl}\, \boldsymbol{i}_3 = f \boldsymbol{i}_2$$

$$\text{(5.4)} \qquad \text{div}\, \boldsymbol{i}_1 = -\frac{\operatorname{tg} \varphi}{\nu} - \frac{f}{\rho} \quad ; \quad \text{div}\, \boldsymbol{i}_2 = 0 \quad ; \quad \text{div}\, \boldsymbol{i}_3 = \mathcal{H} = \frac{1}{\rho} + \frac{1}{\nu}$$

and then, taking into account that

$$\text{(5.5)} \qquad G = \frac{\rho^2 \nu^2 \cos^2 \varphi}{\gamma^2}$$

and the formulae given in (Marussi, 1950) for the Christoffel's symbols and for derivatives of the coefficients of the metric tensor:

$$\text{(5.6)} \qquad \begin{aligned} \text{curl}\, \boldsymbol{v}_1 &= [2 + f(f+g) - f']\, \boldsymbol{i}_2 \\ \text{curl}\, \boldsymbol{v}_2 &= -2 \cos \varphi i_1 - 2 \sin \varphi \boldsymbol{i}_2 \\ \text{curl}\, \boldsymbol{v}_3 &= \frac{1}{\rho\gamma} \left[(1+f^2)(f+g) + ff' + \frac{\rho}{\nu}(2f+h) + \frac{6\,\rho f \omega^2}{\gamma} \right] \boldsymbol{i}_2 \end{aligned}$$

$$\text{(5.7)} \qquad \begin{aligned} \text{div}\, \boldsymbol{v}^1 &= \Delta_2 \varphi = \tau^1 = \frac{1}{\rho} \left(\frac{2f + h - \operatorname{tg} \varphi}{\nu} + \frac{4 f \omega^2}{\gamma} \right) \\ \text{div}\, \boldsymbol{v}^2 &= \Delta_2 \lambda = \tau^2 = 0 \\ \text{div}\, \boldsymbol{v}^3 &= \Delta_2 U = \tau^3 = 2\,\omega^2 \end{aligned}$$

$$\text{(5.8)} \qquad \begin{aligned} \text{div}\, \boldsymbol{v}_1 &= S_{/1} = -\operatorname{tg} \varphi + g + h - f \\ \text{div}\, \boldsymbol{v}_{/2} &= S_2 = 0 \\ \text{div}\, \boldsymbol{v}_3 &= S_{/3} = \frac{-1}{\gamma} \left(2\mathcal{H} + \frac{2\,\omega^2}{\gamma} + \frac{2f^2 - f'}{\rho} - \frac{f \operatorname{tg} \varphi}{\nu} \right) \end{aligned}$$

in which

$$\text{(5.9)} \qquad f = \frac{\partial \lg \gamma}{\partial \varphi} \quad , \quad g = \frac{\partial \lg \rho}{\partial \varphi} \quad , \quad h = \left(1 - \frac{\rho}{\nu}\right) \operatorname{tg} \varphi \quad , \quad f' = \frac{\partial f}{\partial \varphi}$$

are all small quantities, and ω is the angular speed of rotation of the Earth.

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