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The plate on unilateral elastic boundary support. Nota II

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Meccanica dei solidi. — *The plate on unilateral elastic boundary support* (*). Nota II di RAFFAELE TOSCANO (**) e ALDO MACERI (***), presentata (****) dal Corrisp. E. GIANGRECO.

RIASSUNTO. — Si prosegue lo studio del problema della piastra elastica con appoggio elastico unilaterale al bordo, iniziato nella Nota I. Si completa lo studio della unicità della soluzione e si danno risultati di regolarità.

To study the uniqueness of problem (P) solution in β) and γ) cases, we put, for any $v \in W^2(\Omega)$:

$$\Gamma_v^+ = \{x \in \Gamma_E \mid v(x) > 0\} \quad \text{and} \quad \Gamma_v^- = \{x \in \Gamma_E \mid v(x) < 0\},$$

and, for any $p \in P_1 - \{0\}$:

$$\forall x \in \Gamma \qquad \chi_p(x) = \begin{cases} 1, & \text{if} \quad p(x) = 0 \\ 0, & \text{if} \quad p(x) \neq 0 . \end{cases}$$

Let us notice that, if u is solution of problem (P), because:

$$\int_\Gamma E u^+ \, ds = \langle q, \mathbf{1} \rangle > 0,$$

we have $s(\Gamma_u^+) > 0$.

LEMMA 3. *In* β) *and* γ) *cases, if u and $\tilde{u}$ are solutions of problem* (P), *then*:

$$u^+ = \tilde{u}^+ \qquad s\text{-a.e. on } \Gamma_E$$

$$u - \tilde{u} = \tilde{p} \qquad \textit{with} \quad \tilde{p} \in P_1 \qquad \textit{and} \quad \tilde{p}(\xi) = 0 .$$

Proof. From the relations:

$$\sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^s(\tilde{u} - u) D^r(\tilde{u} - u) \, dx + \int_\Gamma E(\tilde{u} - u)(\tilde{u}^+ - u^+) \, ds = 0,$$

$$E(\tilde{u} - u)(\tilde{u}^+ - u^+) \geq 0,$$

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we have:

$$u - \tilde{u} = \tilde{p} \qquad \text{with} \qquad \tilde{p} \in P_1$$

$$E(\tilde{u} - u)(\tilde{u}^+ - u^+) = 0 \qquad s\text{-a.e. on } \Gamma .$$

Hence:

$$\tilde{u}^+ = u^+ \qquad s\text{-a.e. on } \Gamma_E$$

and, consequently:

$$\int_\Gamma E u^+ \tilde{p} \, ds = 0 .$$

Therefore $\tilde{p}(\xi) = 0$, since:

$$\int_\Gamma E u^+ \tilde{p} \, ds = \tilde{p}(\xi) \langle q , \mathbf{1} \rangle . \quad \square$$

Now let us consider the problem:

PROBLEM (Q). *Find* $u \in W^2(\Omega)$ *such that*:

$$\int_\Gamma \chi_p E u^+ \, ds \neq \langle q , \mathbf{1} \rangle \qquad \forall p \in P_1 - \{0\} \qquad \textit{with} \quad p(\xi) = 0 .$$

We will say that problems (P) and (Q) are compatible if they allow a common solution.

THEOREM 6. *If problems* (P) *and* (Q) *are compatible, then problem* (P) *allows a unique solution.*

If problems (P) *and* (Q) *are not compatible, and if* u *is a solution of problem* (P), *denoted as* $\bar{p}$ *an element of* $P_1 - \{0\}$ *such that*:

$$\bar{p}(\xi) = 0 , \qquad \int_\Gamma \chi_{\bar{p}} E u^+ \, ds = \langle q , \mathbf{1} \rangle , \tag{37}$$

all and only solutions of problem (P) *are of the type* $u + \lambda\bar{p}$, *where*:

$$\lambda \in \bigcup_{\substack{\Gamma_0 \subseteq \Gamma_{\bar{p}}^+ \cup \Gamma_{\bar{p}}^- - \Gamma_u^+ \\ s(\Gamma_0)=0}} \left[\sup_{\Gamma_{\bar{p}}^- - (\Gamma_u^+ \cup \Gamma_0)} - u/\bar{p} , \inf_{\Gamma_{\bar{p}}^+ - (\Gamma_u^+ \cup \Gamma_0)} - u/\bar{p} \right]$$

if $s(\Gamma_{\bar{p}}^+) > 0$ *and* $s(\Gamma_{\bar{p}}^-) > 0$,

$$\lambda \in \bigcup_{\substack{\Gamma_0 \subseteq \Gamma_{\bar{p}}^- - \Gamma_u^+ \\ s(\Gamma_0)=0}} \left[\sup_{\Gamma_{\bar{p}}^- - (\Gamma_u^+ \cup \Gamma_0)} - u/\bar{p} , +\infty \right[$$

if $s(\Gamma_{\bar{p}}^{+}) = 0$ *and* $s(\Gamma_{\bar{p}}^{-}) > 0$,

$$\lambda \in \bigcup_{\substack{\Gamma_0 \subseteq \Gamma_{\bar{p}}^{+} - \Gamma_u^{+} \\ s(\Gamma_0)=0}} \left] -\infty \quad , \quad \inf_{\Gamma_{\bar{p}}^{+} - (\Gamma_u^{+} \cup \Gamma_0)} \quad -u/\bar{p} \right]$$

if $s(\Gamma_{\bar{p}}^{+}) > 0$ *and* $s(\Gamma_{\bar{p}}^{-}) = 0$,

λ *is any real number if* $s(\Gamma_{\bar{p}}^{+}) = s(\Gamma_{\bar{p}}^{-}) = 0$.

Proof. Let us suppose that problems (P) and (Q) are compatible, and let u be a common solution. By absurd, let us suppose that problem (P) has a solution $\tilde{u} \neq u$. Putting $\tilde{p} = u - \tilde{u}$ from Lemma 3 we have:

$$\tilde{p} \in P_1 - \{0\} \quad , \quad \tilde{p}(\xi) = 0 \quad , \quad \tilde{p} = 0 \qquad s\text{-a.e. on } \Gamma_u^{+} . \tag{38}$$

From first and second of (38) we have:

$$\int_{\Gamma} \chi_{\tilde{p}} \, Eu^{+} \, ds \neq \langle q , \mathbf{1} \rangle ; \tag{39}$$

moreover because u is solution of (3), we have:

$$\int_{\Gamma} Eu^{+} \, ds = \langle q , \mathbf{1} \rangle . \tag{40}$$

(39) and (40) imply:

$$\int_{\Gamma} (1 - \chi_{\tilde{p}}) \, Eu^{+} \, ds > 0 . \tag{41}$$

On the other hand, taking account of third of (38), we must have:

$$(1 - \chi_{\tilde{p}}) \, Eu^{+} = 0 \qquad s\text{-a.e. on } \Gamma_u^{+} ,$$

and this contrasts with (41).

Let us suppose now that problems (P) and (Q) are not compatible. If u is solution of problem (P), called $\bar{p}$ an element of $P_1 - \{0\}$ satisfying (37), let us at first notice that:

$$\bar{p} = 0 \qquad s\text{-a.e. on } \Gamma_u^{+} . \tag{42}$$

In fact from (37) and because u is solution of (3) results:

$$\int_{\Gamma} (1 - \chi_{\bar{p}}) \, Eu^{+} \, ds = 0 ,$$

which implies:

$$(1-\chi_{\bar{p}})\, Eu^+ = 0 \qquad s\text{-a.e. on } \Gamma_u^+$$

from which (42).

After that, let us analyze the case:

$$s(\Gamma_{\bar{p}}^-) > 0 \quad \text{and} \quad s(\Gamma_{\bar{p}}^+) > 0 .$$

Let $\tilde{u}$ be a solution of problem (P). From (42), taking account of Lemma 3 and that $s(\Gamma_u^+) > 0$, we have the existence of a real number λ such that $\tilde{u} = u + \lambda\bar{p}$. Hence, by Lemma 3:

$$u^+ = (u + \lambda\bar{p})^+ \qquad s\text{-a.e. on } \Gamma_E$$

which implies:

$$u + \lambda\bar{p} \leq 0 \qquad s\text{-a.e. on } \Gamma_{\bar{p}}^+ \cup \Gamma_{\bar{p}}^- - \Gamma_u^+ .$$

Hence a subset Γ_0 of $\Gamma_{\bar{p}}^+ \cup \Gamma_{\bar{p}}^- - \Gamma_u^+$ exists such that $s(\Gamma_0) = 0$ and:

$$\sup_{\Gamma_{\bar{p}}^- - (\Gamma_u^+ \cup \Gamma_0)} - u/\bar{p} \leq \lambda \leq \inf_{\Gamma_{\bar{p}}^+ - (\Gamma_u^+ \cup \Gamma_0)} - u/\bar{p} . \tag{43}$$

Inversely if Γ_0 is a subset of $\Gamma_{\bar{p}}^+ \cup \Gamma_{\bar{p}}^- - \Gamma_u^+$ with zero s-measure, and if λ is a real number satisfying (43), taking account of (42), we have:

$$Eu^+ = E(u + \lambda\bar{p})^+ \qquad s\text{-a.e. on } \Gamma .$$

Then $u + \lambda\bar{p}$ is obviously a solution of (3), i.e. of problem (P).

Similarly we proceed in the other cases. □

Remark 2. Obviously in β) case, if $\forall p \in P_1 - \{0\}$, with $p(\xi) = 0$, $s(\{x \in \Gamma_E \mid p(x) = 0\}) = 0$, problems (P) and (Q) are compatible. On the contrary, by (14), they are incompatible if γ) case is true.

3. – To study regularity of problem (P) solutions, we let for any $\delta > 0$:

$$S_\delta = \{y \in R^n \mid |y| < \delta\} \quad , \quad \Sigma_\delta = \{y \in R^n \mid |y| < \delta , y_n > 0\} \qquad (n \geq 2)$$

$$W(\Sigma_\delta) = \{v \in L^2(\Sigma_\delta) \mid \exists \delta_v \in]0 , \delta[\ni' v(y) = 0 \qquad \text{as} \quad |y| > \delta_v\} .$$

We let also, $\forall i \in \{1 , \cdots , n-1\}$ and $\forall h \in R$:

$$h^i = (0 , \cdots , h , \cdots , 0) \quad , \quad \Sigma_\delta^{i,h} = \{y \in \Sigma_\delta \mid y + h^i \in \Sigma_\delta\}$$

and, if $v \in L^2(\Sigma_\delta)$:

$$\rho_{i,h}\, v(y) = \frac{v(y + h^i) - v(y)}{h} \qquad \text{a.e. on} \quad \Sigma_\delta^{i,h} \ (h \neq 0) .$$

We recall that [3] $\forall v \in W^2(\Sigma_\delta) \cap W(\Sigma_\delta)$:

$$\| v \|_{W^2(\Sigma_\delta)} \leq \text{const.} \left(\sum_{|r|=2} \int_{\Sigma_\delta} | D^r v |^2 \, dy \right)^{\frac{1}{2}} \tag{44}$$

and moreover, if $\overline{\delta} \in]\delta_v, \delta[$ and $h_0 = \min \{\delta - \overline{\delta}, \overline{\delta} - \delta_v\}$:

$$\text{as} \quad 0 < |h| < h_0 \qquad \| \rho_{i,h} v \|_{L^2(\Sigma_{\overline{\delta}})} \leq \left\| \frac{\partial v}{\partial y_i} \right\|_{L^2(\Sigma_\delta)} . \tag{45}$$

Let:

$$b(u, v) = \sum_{\substack{|r|\leq 2 \\ |s|\leq 2}} \int_{\Sigma_\delta} b_{rs} D^s u D^r v \, dy \qquad \forall (u, v) \in (W^2(\Sigma_\delta))^2$$

be a bilinear integro-differential form with $b_{rs} \in L^\infty(\Sigma_\delta)$ and such that:

$$b(v, v) \geq b_0 \sum_{|r|=2} \int_{\Sigma_\delta} | D^r v |^2 \, dy \quad \forall v \in W^2(\Sigma_\delta) \cap W(\Sigma_\delta) \quad (b_0 = \text{const.} > 0); \tag{46}$$

moreover let L be a real-valued linear functional continuous in $W^2(\Sigma_\delta) \cap$ $\cap W(\Sigma_\delta)$ provided with the $W^2(\Sigma_\delta)$ norm and such that:

$$L(\chi) = 0 \qquad \forall \chi \in C_0^\infty(\Sigma_\delta) \tag{47}$$

$$| L(v) | + \left| L\left(\frac{\partial v}{\partial y_j}\right) \right| \leq b' \| v \|_{W^1(\Sigma_\delta)} \qquad \forall v \in C_0^\infty(S_\delta) \tag{48}$$

$$\text{and} \qquad \forall j \in \{1, \cdots, n-1\} \qquad (b' = \text{const.} > 0).$$

THEOREM 7. *If*:

$$\Omega \in \mathbf{R}^{(2),1} \quad , \quad a_{rs} \in C^{0,1}(\overline{\Omega}) \quad , \quad q \in (W^1(\Omega))' ,$$

then any solution u of problem (P) *belongs to* $W^3(\Omega)$ *and we have*:

$$\| u \|_{W^3(\Omega)} \leq \gamma \left(\| q \|_{(W^1(\Omega))'} + \| E \|_{L^\infty(\Gamma)} \cdot \| u^+ \|_{L^2(\Gamma)} + \| u \|_{W^2(\Omega)} \right)$$

where $\gamma = \gamma(a_{rs}, \Omega)$.

Proof. We proceed as in [3] (Theorem 2), taking account of (1) and (2). □

Remark 3. In the hypotheses of Theorem 7, because $W^3(\Omega) \subseteq C^{1,\lambda}(\overline{\Omega})$ $\forall \lambda \in]0, 1[$, any problem (P) solution has first order partial derivatives satisfying in $\overline{\Omega}$ a Hölder condition with any exponent $\lambda \in]0, 1[$.

The following lemma allows us to state a further regularity result.

LEMMA 4. *Let* $b_{rs} \in C^{1,1}(\overline{\Sigma}_\delta)$. *If* $u \in W^2(\Sigma_\delta)$ *is such that*:

$$|b(u, v) + L(v)| \leq b'' \|v\|_{L^2(\Sigma_\delta)} \tag{49}$$

$$\forall v \in W^2(\Sigma_\delta) \cap W(\Sigma_\delta) \qquad (b'' = const. > 0)$$

then, for any $\delta' \in]0, \delta[$:

$$u \in W^4(\Sigma_{\delta'}) \quad , \quad \|u\|_{W^4(\Sigma_{\delta'})} \leq \gamma (b' + b'' + \|u\|_{W^2(\Sigma_\delta)})$$

where $\gamma = \gamma(\delta', b_{rs})$.

Proof. At first from (48) and (49) we have:

$$|b(u, v)| \leq (b' + b'') \|v\|_{W^1(\Sigma_\delta)} \qquad \forall v \in W^2(\Sigma_\delta) \cap W(\Sigma_\delta),$$

and hence ([3], Lemma 4), for any $\delta' \in]0, \delta[$:

$$u \in W^3(\Sigma_{\delta'}), \qquad \|u\|_{W^3(\Sigma_{\delta'})} \leq \gamma(b' + b'' + \|u\|_{W^2(\Sigma_\delta)}) \tag{50}$$

$$\text{with} \quad \gamma = \gamma(\delta', b_{rs}).$$

After that, let $\delta' < \delta_2 < \delta_3 < \delta_4 < \delta$, $\eta \in C_0^\infty(S_{\delta_3})$ with $\eta = 1$ on $\overline{S}_{\delta_2}$, $\tilde{u} = u\eta$, $\delta_\eta < \delta'' < \delta_3$, $h_0 = \min\{\delta - \delta_4, \delta_4 - \delta_3, \delta_3 - \delta'', \delta'' - \delta_\eta\}$. Let us prove that:

$$\text{as} \quad |r| = 2 \qquad \text{and as} \quad i, j \in \{1, \cdots, n-1\} \tag{51}$$

$$\frac{\partial}{\partial y_i}\left(\frac{\partial}{\partial y_j} D^r \tilde{u}\right) \in L^2(\Sigma_\delta),$$

$$\left\|\frac{\partial}{\partial y_i}\left(\frac{\partial}{\partial y_j} D^r \tilde{u}\right)\right\|_{L^2(\Sigma_\delta)} \leq \text{const.}\,(b' + b'' + \|u\|_{W^2(\Sigma_\delta)}).$$

Given a sequence $\{v_n\}$ of elements of $C_0^\infty(S_{\delta_3})$ such that:

$$\lim_{n\to+\infty} \left\|v_n - \rho_{i,h}\left(\frac{\partial \tilde{u}}{\partial y_j}\right)\right\|_{W^2(\Sigma_\delta)} = 0, \tag{52}$$

we have, $\forall n \in N$ and as $0 < |h| < h_0$:

$$b\left(\rho_{i,h}\left(\frac{\partial \tilde{u}}{\partial y_j}\right), v_n\right) = b\left(u, \eta \cdot \rho_{i,-h}\left(\frac{\partial v_n}{\partial y_j}\right)\right) +$$

$$+ \sum_{\substack{|r|\leq 2 \\ |s|\leq 2}} \sum_{\substack{\alpha \leq r \\ \alpha \neq (0,\cdots,0)}} \binom{r}{\alpha} \int_{\Sigma_{\delta_3}} \frac{\partial}{\partial y_j}(b_{rs}(y) D^s u(y) \cdot$$

$$\cdot D^\alpha \eta(y)) \cdot D^{r-\alpha}(\rho_{i,-h} v_n(y))\, dy - \sum_{\substack{|r|\leq 2 \\ |s|\leq 2}} \int_{\Sigma_{\delta_3}} D^r v_n(y) \cdot$$

$$\cdot \frac{\partial}{\partial y_j} \left(\frac{b_{rs}(y+h^i) - b_{rs}(y)}{h} \eta(y+h^i) D^s u(y+h^i) \right) dy +$$

$$+ \sum_{\substack{|r|\le 2 \\ |s|\le 2}} \sum_{\substack{\alpha \le s \\ \alpha \ne (0,\cdots,0)}} \binom{s}{\alpha} \int_{\Sigma_{\delta_3}} D^r v_n \cdot$$

$$\cdot \frac{\partial}{\partial y_j} (b_{rs} \rho_{i,h} (D^\alpha \eta D^{s-\alpha} u)) dy - \sum_{\substack{|r|\le 2 \\ |s|\le 2}} \int_{\Sigma_{\delta_3}} \frac{\partial b_{rs}}{\partial y_j} \cdot$$

$$\cdot D^s \rho_{i,h} (\eta u) D^r v_n dy .$$

Hence, because:

$$b\left(\rho_{i,h}\left(\frac{\partial \tilde{u}}{\partial y_j}\right), v_n\right) = b\left(\rho_{i,h}\left(\frac{\partial \tilde{u}}{\partial y_j}\right), v_n\right) +$$

$$+ b\left(u, \eta\rho_{i,-h}\left(\frac{\partial v_n}{\partial y_j}\right)\right) + L\left(\eta\rho_{i,-h}\left(\frac{\partial v_n}{\partial y_j}\right)\right) -$$

$$- \left(b\left(u, \eta\rho_{i,-h}\left(\frac{\partial v_n}{\partial y_j}\right)\right) + L\left(\eta\rho_{i,-h}\left(\frac{\partial v_n}{\partial y_j}\right)\right)\right),$$

we have, taking account of (48), (49), (50) and (45):

$$\left| b\left(\rho_{i,h}\left(\frac{\partial \tilde{u}}{\partial y_j}\right), v_n\right) \right| \le \text{const.} \, (b' + b'' + \|u\|_{W^2(\Sigma_\delta)}) \cdot \|v_n\|_{W^2(\Sigma_\delta)}$$

from which, taking account of (52), (44) and (46), we have (51).

Because (47) is true, we can complete the Proof in the same way already used in [3] (Lemma 5). □

THEOREM 8. *If*:

$$\Omega \in \mathbf{R}^{(3),1} \quad , \quad a_{rs} \in C^{1,1}(\overline{\Omega}) \quad , \quad q \in L^2(\Omega) \quad , \quad E \in W^1(\Gamma) ,$$

then any solution u *of problem* (P) *belongs to* $W^4(\Omega)$ *and we have*:

$$\|u\|_{W^4(\Omega)} \le \gamma (\|q\|_{L^2(\Omega)} + \|E\|_{W^1(\Gamma)} \|u^+\|_{C^{0,1}(\Gamma)} + \|u\|_{W^2(\Omega)})$$

where $\gamma = \gamma(a_{rs}, \Omega)$.

Proof. At first we have [4]:

a) For any open Ω' with closure contained in Ω, results:

$$u \in W^4(\Omega') \quad , \quad \|u\|_{W^4(\Omega')} \le \gamma (\|q\|_{L^2(\Omega)} + \|u\|_{W^2(\Omega)})$$

where $\gamma = \gamma(a_{rs}, \Omega')$.

Now we fix on Γ a point $\bar{x}$. Because $\Omega \in \mathbf{R}^{(3),1}$, an open neighbourhood U of $\bar{x}$, a $\delta > 0$ and an one-to-one transformation $\Phi = (\Phi_1, \Phi_2)$ mapping S_δ onto U exist such that:

$$\Phi \in C^{3,1}(S_\delta) \quad , \quad \Phi^{-1} \in C^{3,1}(U) \quad , \quad \left| \frac{\partial(\Phi_1, \Phi_2)}{\partial(y_1, y_2)}(y) \right| = 1 \qquad \forall y \in S_\delta ,$$

$$\Phi(\Sigma_\delta) = U^+ \qquad \text{where} \quad U^+ = \Omega \cap U .$$

Let us, for any $(\tilde{v}, \tilde{w}) \in (W^2(\Sigma_\delta))^2$ [resp. $\tilde{v} \in W^2(\Sigma_\delta) \cap W(\Sigma_\delta)$] $v = \tilde{v} \circ \Phi^{-1}$ and $w = \tilde{w} \circ \Phi^{-1}$ $\left[\text{resp. } v = \begin{cases} \tilde{v} \circ \Phi^{-1} & \text{on } U^+ \\ 0 & \text{on } \Omega - U^+ \end{cases} \right]$ and consider the sum [resp. the function]:

$$(53) \qquad \sum_{\substack{|r|=2 \\ |s|=2}} \int_{U^+} a_{rs}(x)\, D^s v(x)\, D^r w(x)\, dx \left[\text{resp. } L(\tilde{v}) = \int_\Gamma E u^+ v\, ds \right] .$$

Replacing x by $\Phi(y)$ and writing $(D^s v)(\Phi(y))$ and $(D^r w)(\Phi(y))$ in terms of derivatives of $\tilde{v}$ and $\tilde{w}$ respectively, (53) becomes:

$$\sum_{\substack{|\alpha| \le 2 \\ |\beta| \le 2}} \int_{\Sigma_\delta} b_{\alpha\beta}(y)\, D^\beta \tilde{v}(y)\, D^\alpha \tilde{w}(y)\, dy ,$$

where $b_{\alpha\beta} \in C^{1,1}(\bar{\Sigma}_\delta)$ and depends on a_{rs} and Φ only.

Obviously the bilinear integro-differential form:

$$b(\tilde{v}, \tilde{w}) = \sum_{\substack{|\alpha| \le 2 \\ |\beta| \le 2}} \int_{\Sigma_\delta} b_{\alpha\beta}(y)\, D^\beta \tilde{v}(y)\, D^\alpha \tilde{w}(y)\, dy \qquad \forall (\tilde{v}, \tilde{w}) \in (W^2(\Sigma_\delta))^2$$

satisfies (46). It is also obvious that the functional L verifies (47) and (48). Then, we put $\tilde{u} = u \circ \Phi$ and note as $\tilde{v}$ an element of $W^2(\Sigma_\delta) \cap W(\Sigma_\delta)$. Because:

$$\sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs}\, D^s u\, D^r v\, dx + \int_\Gamma E u^+ v\, ds = \int_\Omega q v\, dx ,$$

we have:

$$| b(\tilde{u}, \tilde{v}) + L(\tilde{v}) | \le \| q \|_{L^2(\Omega)} \| \tilde{v} \|_{L^2(\Sigma_\delta)} .$$

Consequently, by Lemma 4:

$$\tilde{u} \in W^4(\Sigma_{\delta'}) \quad , \quad \| \tilde{u} \|_{W^4(\Sigma_{\delta'})} \le \gamma(\Phi, a_{rs}, \delta') \cdot$$

$$\cdot (\| q \|_{L^2(\Omega)} + \| E \|_{W^1(\Gamma)} \| u^+ \|_{C^{0,1}(\Gamma)} + \| u \|_{W^2(\Omega)})$$

for any $\delta' \in]0, \delta[$. Hence, putting $U' = \Phi(S_{\delta'})$ and $U'^{+} = U' \cap \Omega$, we have:

$$u \in W^4(U'^{+}) \quad , \quad \| u \|_{W^4(U'^{+})} \leq \gamma(\Phi, a_{rs}, \delta') \cdot$$

$$\cdot (\| q \|_{L^2(\Omega)} + \| E \|_{W^1(\Gamma)} \cdot \| u^{+} \|_{C^{0,1}(\Gamma)} + \| u \|_{W^2(\Omega)}) .$$

The regularization at the boundary is acquired and therefore, taking account of *a*), the thesis is proven. □

Remark 4. In the hypotheses of Theorem 8, because $W^4(\Omega) \subseteq C^{2,\lambda}(\overline{\Omega})$ $\forall \lambda \in]0, 1[$, any solution u of problem (P) has second order partial derivatives satisfying in $\overline{\Omega}$ a Hölder condition with any exponent $\lambda \in]0, 1[$.

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