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Analisi matematica. — *Results on linear and nonlinear hyperbolic boundary value problems at resonance.* Nota di MICHAEL W. SMILEY (*), presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — Si considera l'equazione non lineare nell'incognita $u(t, x)$ ((1, 1) del testo) soddisfatta in un cilindro $G = (0, T) \times \Omega$ (Ω dominio limitato di R^n) con condizioni al contorno tipo Dirichlet o Neumann sulla superficie laterale di G e con relazioni omogenee fra u e u_t sulle basi.

Si stabiliscono per la (1) e nel caso di risonanza alcuni teoremi di perturbazione.

§ 1. INTRODUCTION

We consider nonlinear hyperbolic boundary value problems in which the partial differential equation takes the form

$$\frac{\partial^2 u}{\partial t^2} + A(t)\, u = \varepsilon [g(u) + f] . \tag{1.1}$$

This equation is to be satisfied weakly, in the cylinder $G = (0, T) \times \Omega$ where Ω is an open bounded subset of $\mathbf{R}^n$. In equation (1.1) we assume that $A(t)$ is a strongly elliptic operator, uniformly in Ω, having order $2m$ for $m \geq 1$; that $g(\cdot)$ is a nonlinear Nemytsky operator generated by the real-valued function $g : \mathbf{R} \to \mathbf{R}$; and that ε is a real parameter. In addition to satisfying (1.1), a solution must also satisfy boundary conditions, possibly of mixed type, on the lateral surface and ends of the cylinder $G = (0, T) \times \Omega$. The boundary conditions are also to be satisfied in a weak sense. On the lateral surface $(0, T) \times \partial\Omega$ either Dirichlet or Neumann conditions will be in force. On the ends of the cylinder, $\{0\} \times \Omega$ and $\{T\} \times \Omega$, we will impose linear homogeneous two point boundary condition of the form

$$\begin{aligned} B_1 u &= a_{11}\, u(0, x) + a_{12}\, u_t(0, x) + b_{11}\, u(T, x) + b_{12}\, u_t(T, x) = 0 \\ B_2 u &= a_{21}\, u(0, x) + a_{22}\, u_t(0, x) + b_{21}\, u(T, x) + b_{22}\, u_t(T, x) = 0 , \end{aligned} \tag{1.2}$$

where $(a_{11}, a_{12}, b_{11}, b_{12})$ and $(a_{21}, a_{22}, b_{21}, b_{22})$ are linearly independent constant vectors in $\mathbf{R}^4$. Observe that the Dirichlet, Neumann, and periodic boundary condition are particular instances of (1.2).

The phenomena of mathematical resonance is exhibited by many of the boundary value problems falling into the class of problems described above.

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By resonance we mean that the corresponding linear homogeneous boundary value problem ($\varepsilon = 0$) has a nontrivial subspace of solutions. In contrast to the elliptic case, we often find that the subspace of solutions to the homogeneous problem is an infinite dimensional space. This feature is of particular interest in our work.

We have followed Lions [7] and Lions-Magenes [8] in taking the viewpoint that our partial differential equation is an ordinary differential equation having meaning in a Hilbert space. Thus we consider mappings $u : (0, T) \to V$ where V is a real Hilbert space of functions depending only on $x \in \Omega$. In this way the spatial boundary conditions on the lateral surface $(0, T) \times \partial\Omega$ will be enforced by requiring $u(t) \in V$ (a.e.) $t \in (0, T)$, while we may equivalently replace (1.2) by

$$
\begin{aligned}
B_1 u &= a_{11} u(0) + a_{12} u'(0) + b_{11} u(T) + b_{12} u'(T) = 0 \\
B_2 u &= a_{21} u(0) + a_{22} u'(0) + b_{21} u(T) + b_{22} u'(T) = 0 .
\end{aligned}
\tag{1.3}
$$

Boundary conditions (1.3) will then be equations in a Hilbert space. In this setting we formulate a weak problem which properly generalizes the classical problem. This formulation also generalizes the weak Cauchy problem of Lions-Magenes [8, p. 265].

We have followed Cesari [1] and Hale [3] in dealing with the difficult problem at resonance. After developing the necessary linear theory, we show that many of these problems fit into the well-known framework of alternative problems. We are then able to prove theorems on nonlinear perturbations for these boundary value problems. The main theoretical tool used is the implicit function theorem in Banach spaces. This is in contrast to the elliptic case in which the Euclidean space version of the implicit function theorem can be used. This reflects the possible infinite dimensionality of the kernel for the problem.

§ 2. A Statement of Results

Before stating our results we establish some notations. We will restrict our attention to the situation in which $H = L^2(\Omega)$ and $V \subset H$ is a Hilbert space of real-valued functions. We assume that V is dense in H. Let us denote by $L^2(0, T; H)$ and $L^2(0, T; V)$ the Hilbert spaces of norm square integrable functions from $(0, T) \to H$ and V respectively (cf. Dunford-Schwartz [2]). We define the Hilbert space $W(0, T)$ by (cf. Lions-Magenes [8])

$$
W(0, T) = \left\{ u \in L^2(0, T; V) : \frac{du}{dt} \in L^2(0, T; H) \right\}
$$

where the derivative is a weak derivative (cf. Schwartz [11]). We also define some subspaces of $W(0, T)$. Let $W_0(0, T)$ be the closure in $W(0, T)$ of

$C_0^\infty(0, T; V)$, the set of C^∞ functions from $(0, T)$ into V having compact support in $(0, T)$. Let $W_{per}(0, T)$ be the closure in $W(0, T)$ of $C_{per}^\infty(0, T; V)$, the set of C^∞ functions from $\mathbf{R}$ into V which are T-periodic. Finally we let $W^\infty(0, T)$ denote the Banach space

$$W^\infty(0, T) = \left\{ u \in L^\infty(0, T; V) : \frac{du}{dt} \in L^\infty(0, T; H) \right\}$$

where again the derivative is a weak derivative. As a norm in $W(0, T)$, $W_0(0, T)$, and $W_{per}(0, T)$ we will use

$$\| u \|_W^2 = \| u \|_{L^2(0,T;V)}^2 + \left\| \frac{du}{dt} \right\|_{L^2(0,T;H)}^2 ,$$

while in $W^\infty(0, T)$ we will use

$$\| u \|_{W^\infty} = \| u \|_{L^\infty(0,T;V)} + \left\| \frac{du}{dt} \right\|_{L^\infty(0,T;H)} .$$

Let $A : V \to V^*$, the dual of V, be a continuous linear mapping and associate to A the continuous bilinear form on V given by $a(u, v) = (Au, v)$ $\forall u, v \in V$. The bracket on the right represents the dual action of Au on v. We assume that $a(u, v) = a(v, u)$ for all $u, v \in V$ and that there are numbers $\lambda, \alpha \in \mathbf{R}$ with $\alpha > 0$ such that

$$a(u, u) + \lambda \| u \|_H^2 \geq \alpha \| u \|_V^2 \qquad \forall u \in V .$$

In addition we assume that there is a complete orthonormal basis, $\{w_i\}$, for V consisting of eigenfunctions for the operator A. We further assume that $\{w_i\}$ is an orthogonal set in H. Let $\{\lambda_i\}$ be the set of corresponding eigenvalues (not assumed to be distinct). We assume that $\lambda_i \geq 0$ for all $i = 1, 2, \cdots$.

Consider the following linear problem

$$\frac{d^2 u}{dt^2} + Au = f , \qquad 0 < t < T, \tag{2.1}$$

$$B_1 u = B_2 u = 0 , \tag{2.2}$$

where $B_1 u = B_2 u = 0$ represents either Dirichlet, Neumann, or periodic boundary conditions. Let W denote $W_0(0, T)$, $W(0, T)$, or $W_{per}(0, T)$ if the boundary conditions are Dirichlet, Neumann, or periodic respectively.

We define a continuous bilinear form on W by setting

$$B(u, w) = \int_0^T [-(u', w')_H + a(u, w)] \, dt \qquad u, w \in W.$$

We say that u is a weak solution to (2.1)–(2.2) if $u \in W$ and

$$B(u, w) = (f, w)_{L^2(0,T;H)} \qquad \forall w \in W. \tag{2.3}$$

If f is identically zero then we have the linear homogeneous problem

$$B(u, w) = 0 \qquad \forall w \in W. \tag{2.4}$$

Let $X_0 = \{u \in W : u \text{ satisfies } (2.4)\}$. Clearly X_0 is a closed subspace of W. Note that we have the case of resonance exactly when dimension $(X_0) \geq 1$.

Let Σ denote the point spectrum of the corresponding scalar problem

$$\frac{d^2 \phi}{dt^2} + \lambda\phi = 0, \qquad 0 < t < T, \tag{2.5}$$

$$B_1 \phi = B_2 \phi = 0, \tag{2.6}$$

and set $\sigma_i = \text{dist}(\lambda_i, \Sigma)$.

THEOREM 1. a) *Let $f \in L^2(0, T; H)$ and assume that all but a finite number of the λ_i are in Σ. There is a solution of problem* (2.1)–(2.2) *if and only if*

$$(f, u_0)_{L^2(0,T;H)} = 0 \qquad \forall u_0 \in X_0. \tag{2.7}$$

If f satisfies the above condition then there is a unique solution u_1 with the property that $u_1 \in X_0^{\perp}$. Here the orthogonality is taken with respect to the inner product in W. *In addition, there is a constant $c > 0$ such that* $\| u_1 \|_{W^\infty} \leq c \| f \|_{L^2(0,T;H)}$.

b) *Let $f \in L^2(0, T; H)$ and assume that f has a weak derivative* $\frac{df}{dt} \in L^2(0, T; H)$.

If there is a number $\sigma_0 > 0$ such that for all $i = 1, 2, 3, \cdots$ either $\lambda_i \in \Sigma$ or $\sigma_i \geq \sigma_0$ then there is a solution of problem (2.1)–(2.2) *if and only if f satisfies* (2.7). *If f satisfies this condition then there is a unique solution u_1 with the property that $u_1 \in X_0^{\perp}$ and a constant $c > 0$ such that*

$$\| u_1 \|_{W^\infty} \leq c \left(\| f \|^2_{L^2(0,T;H)} + \left\| \frac{df}{dt} \right\|^2_{L^2(0,T;H)} \right)^{1/2}.$$

We now consider a nonlinear Nemytsky operator $g : W(0, T) \to L^2(0, T; H)$ generated by the real-valued function $g : \mathbf{R} \to \mathbf{R}$. Let $f \in L^2(0, T; H)$ satisfy (2.7). We pose the nonlinear boundary value problem

$$\frac{d^2 u}{dt^2} + Au = \varepsilon [g(u) + f], \qquad 0 < t < T, \tag{2.8}$$

$$B_1 u = B_2 u = 0, \tag{2.9}$$

where $B_1 u = B_2 u = 0$ again denotes either Dirichlet, Neumann, or periodic boundary conditions. We say that u is a weak solution of (2.8)–(2.9) if $u \in W$, where $W = W_0(0, T)$, $W(0, T)$, or $W_{per}(0, T)$ as before according to the boundary conditions, and

$$B(u, w) = \epsilon\,(g(u) + f, w)_{L^2(0,T;H)} \qquad \forall w \in W. \tag{2.10}$$

THEOREM 2. *We assume that* $g: W^\infty(0, T) \to W^\infty(0, T)$ *is continuously differentiable as a nonlinear map from any open ball about the origin into* $W^\infty(0, T)$. *If, as a real-valued function,* $g(0) = 0$ *and* $g'(0) \neq 0$, *then for* ϵ_0 *sufficiently small there is a unique local family of weak solutions,* $\{u(\epsilon) : |\varepsilon| < \varepsilon_0\}$ *of problem* (2.8)–(2.9). *Moreover the mapping* $\varepsilon \mapsto u(\varepsilon)$ *is a continuous mapping with* $u(0) = 0$ *being the zero solution to the homogeneous problem* (2.4).

Remarks. 1) We remark that there are two different types of assumptions imposed on g. One deals with its properties as a real-valued function while the other deals with its smoothness as a map in Banach space. The later condition can be reduced to a smoothness requirement on the real-valued function g once V has been specified.

2) The arguments used to establish the above results are easily modified to obtain previous results [4], [5], [6], [9], [10], [12] as special cases of the above situation.

As an example we mention that Theorem 2 can be applied to the problem

$$\begin{aligned} u_{tt} - u_{xx} &= \varepsilon\,[g(u) + f], && (t, x) \in \mathbf{R} \times (0, \pi), \\ u(t, 0) &= u(t, \pi) = 0, && t \in \mathbf{R}, \\ u(t + 2\pi, x) &= u(t, x), && (t, x) \in \mathbf{R} \times (0, \pi), \end{aligned}$$

where we assume that $g \in C^3(\mathbf{R})$, $g(0) = 0$, $g'(0) \neq 0$ and that $f \in L^2(G)$, $G = (0, 2\pi) \times (0, \pi)$, is orthogonal to every solution of the homogeneous problem. A modification of Theorem 2, in which the orthogonality condition is removed, can be proven provided that g is globally monotone. The details of this and proofs of Theorems 1 and 2 will be given in forthcoming papers.

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