Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Generation of analytic semigroups by elliptic operators of second order in Hölder spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **69** (1980), n.6, p. 324–326.

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<http://www.bdim.eu/item?id=RLINA_1980_8_69_6_324_0>

Analisi matematica. — *Generation of analytic semigroups by elliptic operators of second order in Hölder spaces.* Nota di SERGIO CAMPANATO, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Nella presente nota si comunica il seguente risultato:
Un operatore ellittico del secondo ordine, con condizione di Dirichlet al bordo, è generatore infinitesimale di un semigruppo analitico in t con la topologia degli spazi Hölderiani.
La dimostrazione sarà esposta nel lavoro [2].

On an open bounded subset $\Omega \subset \mathrm{R}^n$, with boundary $\partial\Omega$ of class C^2, we consider the operator

$$\mathrm{E}u = -\sum_{ij=1}^{n} \mathrm{D}_i (a_{ij} \mathrm{D}_j u) + \sum_{i=1}^{n} b_i \mathrm{D}_i u \tag{1}$$

where $\mathrm{D}_i = \partial/\partial x_i$, a_{ij} are real valued functions, b_i and u are complex valued functions.

The operator E will have coefficients a_{ij} and b_i subject to the following assumptions:

(I) $a_{ij} \in \mathrm{C}^0(\overline{\Omega})$ and satisfy the ellipticity condition

$$\sum_{ij} a_{ij}(x)\, \xi_j\, \xi_i \geq \nu \sum_i \xi_i^2 , \qquad \nu > 0 \tag{2}$$

for all $x \in \overline{\Omega}$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathrm{R}^n$;

(II) $b_i \in \mathrm{L}^\infty(\Omega)$, i.e. they are measurable and bounded functions on Ω.

The Sobolev space $\mathrm{H}_0^1(\Omega)$ is the completion in the norm

$$|u|_{1,\Omega} = \left(\int_\Omega \sum_i |\mathrm{D}_i u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \tag{3}$$

of the space $\mathrm{C}_0^\infty(\Omega)$.

We set

$$a(u, \varphi) = \int_\Omega \sum_{ij} a_{ij} \mathrm{D}_j u \mathrm{D}_i \overline{\varphi} + \sum_i b_i \mathrm{D}_i u \overline{\varphi} \,\mathrm{d}x$$

and

$$\lambda = \frac{1}{2\nu} \sup_\Omega \sum_i |b_i(x)|^2 .$$

(*) Nella seduta del 6 dicembre 1980.

It is well known that, for all $u \in H_0^1(\Omega)$,

$$\mathcal{R}e\ a(u, u) \geq \frac{\nu}{2} |u|^2_{1,\Omega} - \lambda \int_\Omega |u|^2 dx, \tag{4}$$

therefore, if μ is a complex parameter with $\mathcal{R}e\ \mu > \lambda$ and $f \in L^2(\Omega)$, there exists a unique solution u of the Dirichlet problem

$$\begin{cases} u \in H_0^1(\Omega) \\ \mu \int_\Omega u\bar{\varphi}\, dx + a(u, \varphi) = \int_\Omega f\bar{\varphi}\, dx, \qquad \forall \varphi \in H_0^1(\Omega) \end{cases} \tag{5}$$

and, as it is known, the following estimate holds

$$\left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \leq \frac{K}{|\mu| - \lambda}\left(\int_\Omega |f|^2 dx\right)^{\frac{1}{2}}, \tag{6}$$

where the constant K does not depend on μ and λ.

Also the following smoothness results are well known

$$f \in L^p(\Omega),\ p > 1,\ \Rightarrow u \in L^p(\Omega), \tag{7}$$

$$f \in C^0(\overline{\Omega}) \Rightarrow u \in C^0(\overline{\Omega}), \tag{8}$$

$$f \in C^{0,\alpha}(\overline{\Omega}) \Rightarrow u \in C^{0,\alpha}(\overline{\Omega}), \tag{9}$$

where $C^0(\overline{\Omega})$ is the space of continuous functions on $\overline{\Omega}$, normed as usual, and $C^{0,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, is the space of functions $u : \overline{\Omega} \Rightarrow \mathbf{C}$ such that

$$[u]_{\alpha,\overline{\Omega}} = \sup_{x,y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha} < +\infty$$

equipped with the norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = [u]_{\alpha,\overline{\Omega}} + \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}. \tag{10}$$

However, only in cases (7) and (8) estimates analogous to (6) were known, where the L^2-norms are obviously substituted by the L^p and $C^0(\overline{\Omega})$-norms respectively (for case (8) see for instance [3]. Recently H. B. Stewart extended his result in [3] also to the case of general boundary conditions [4]).

But up to now it was not known whether there was an estimate analogous to (6) also for Hölder norms.

Recently I obtained the following result:

THEOREM 1. *There is a* $\lambda_0 > 0$ *such that, if* $\mathcal{R}e\, \mu > \lambda_0$, $f \in C^{0,\alpha}(\overline{\Omega})$ *and* f *vanishes on* $\partial\Omega$, *the solution* u *of problem* (5) *satisfies the following estimate*

$$\| u \|_{C^{0,\alpha}(\overline{\Omega})} \leq \frac{K}{|\mu| - \lambda_0} \| f \|_{C^{0,\alpha}(\overline{\Omega})}, \tag{11}$$

with a constant K *independent of* μ *and* λ_0.

Note that, without the assumption that $f = 0$ on $\partial\Omega$, estimate (11) does not hold [5].

As it is known, inequality (11) is linked with the problem of knowing whether the elliptic operator E generates an analytic semigroup in the Hölder topology.

Estimate (11) is obtained by adapting to the case $\mu \neq 0$ the technique developed in [1] for the case $\mu = 0$. The proof is contained in the forthcoming paper [2].

I believe that the same approach can be used also for more general elliptic operators and boundary conditions.

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