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Projection of Wreath Products of Lie Algebras

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Algebra. — *Projection of Wreath Products of Lie Algebras.* Nota di ALEXANDER A. LAŠHI (*), presentata (**) dal Socio G. ZAPPA.

RIASSUNTO. — Sia K un dominio a ideali principali, che non sia un campo e sia L un'algebra di Lie su K. Principale risultato: Se $L = A\,wr\,B$, cioè se L è prodotto intrecciato delle algebre di Lie precarie A e B, ogni isomorfismo reticolare normale di L è indotto da un isomorfismo.

Si prova anche, mediante un esempio, che per le algebre sopra un campo il teorema non è vero.

Let K be a commutative ring which contains an identity and no zero divisors. These properties of K will not always be explicitly stated in what follows; moreover, additional conditions on K will sometimes be specified.

Let L be a Lie algebra over K; S (L) is the lattice of all subalgebras of L; $\varphi : S(L) — S(L^\varphi)$ will denote a lattice isomorphism.

The aim of this article is to study the lattice isomorphisms (projections) of a wreath product of Lie algebras. We will make use of generally accepted terminology and notation (see, for example, [1], [2], [3]).

A Lie algebra L over K is called proper if L is torsion-free as a K-module.

The lattice isomorphism $\varphi : S(L) — S(L^\varphi)$ is called *normal* if $N(A)^\varphi = N(A^\varphi)$, for each subalgebra $A \subseteq L$.

Let V be a variety of Lie algebras over K, A an algebra from V and B some algebra. A Lie algebra W is called *wreath* V *-product* of the algebras A and B (denoted by $W = A\,wr_V\,B$) if the following three conditions hold:

(*i*) $W = \mathrm{alg}(A, B)$.

(*ii*) Let $k \leq \mathrm{id}_W(A)$ be the ideal of W generated by A. Then $k \in V$.

(*iii*) If $\varphi : A \to C$, $\psi : B \to C$ are two homomorphisms of A and B into the Lie algebra C over K such that

(*a*) $C = \mathrm{alg}(\varphi(A), \psi(B))$,

(*b*) the ideal $S = \mathrm{id}_C(\varphi(A)) \in V$,

then there exists a homomorphism $\mu : W \to C$ such that

$$\mu|_A = \varphi \qquad \text{and} \qquad \mu|_B = \psi,$$

were $\mu|_A$ and $\mu|_B$ are restrictions of the homomorphism μ on subalgebras A and B, respectively.

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In the case when $V = \mathfrak{A}$ is the variety of abelian algebras W is called simply *wreath product* and denoted by $W = A wr B$. For the definition of wreath product see [2], [4].

PROPOSITION 1. (*see*, [2]). *Let* $W = A\, wr_V\, B$, *then* W *is a split extension of* B *by* $k = \mathrm{id}_W(A)$.

PROPOSITION 2. *If* $W = A wr_V B$ *and* B_1 *is a subalgebra of* B *then*

$$W_1 = \mathrm{alg}\,(A , B_1) = A\, wr_V\, B_1 .$$

PROPOSITION 3. *Let* $W = A\, wr\, B$ *and* A_1 , B_1 *be subalgebras of* A *and* B, *respectively*, *then*

$$W_1 = \mathrm{alg}\,(A_1 , B_1) = A_1\, wr\, B_1 .$$

Making use of these facts, by the theorem 2 from [3] and supposing that K is a principal ideal domain, which is not a field, one may prove:

THEOREM. *If the Lie algebra* $W = A\, wr\, B$ *is a wreath product of proper Lie algebras* A *and* B, *then every normal lattice isomorphism of* W *is induced by an isomorphism.*

Now we construct an example which shows that for the algebras over a field the theorem is not true.

Example. Let P be a field of p elements (p is a prime number) and L a Lie algebra over P.

$$L = \mathrm{alg}\,(x)\, wr\, \mathrm{alg}\,(b) = \mathrm{alg}\,(b)\, \lambda \prod_{i=0}^{\infty} \mathrm{alg}\,(x_i)\, , \quad (\mathrm{ad}\,(b))\,(x_i) = x_{i+1}$$
$$(i = 0 , 1 , 2 , \cdots) .$$

Each element l of L has an unique expression in the form

$$l = \mu b + \sum_{i=0}^{h(l)} \mu_i x_i \qquad (\mu , \mu_i \in P) .$$

Define an one-to-one relation $f : L \rightarrow L$ as follows:

$$f(l) = \begin{cases} \displaystyle\sum_{i=0}^{h(l)} \mu_i x_i , & \text{if } \mu = 0 , \\ \displaystyle\mu b + \sum_{i=0}^{h(l)} \mu_i x_i + \sum_{i=0}^{h(l)} \mu_i x_{i+1} , & \text{if } \mu \neq 0 \end{cases}$$

and let us show that f induces autoprojection of L i.e., that for each $l_1 , l_2 \in L$

$$f(l_1 + l_2) , f([l_1 , l_2]) \in \mathrm{alg}\,(f(l_1) , f(l_2)).$$

If $l_1 , l_2 \in X = \prod_{i=0}^{\infty} \mathrm{alg}\,(x_i)$, then it is clear, because

$$0 = f([l_1 , l_2]) = [f(l_1) , f(l_2)] , f(l_1 + l_2) = f(l_1) + f(l_2) \in \mathrm{alg}\,(f(l_1) , f(l_2)).$$

Let

$$l_1 = \mu B + \Sigma \mu_i x_i \bar{\in} X , \; l_2 = \Sigma \alpha_i x_i \in X .$$

Then

$$f([l_1, l_2]) = f([\mu b + \Sigma \mu_i x_i , \Sigma \alpha_i x_i]) = f(\mu \Sigma \alpha_i x_{i+1}) = \mu \Sigma \alpha_i x_{i+1}$$

$$[f(l_1), f(l_2)] = f([\mu b + \Sigma \mu_i x_i + \Sigma \mu_i x_{i+1} , \Sigma \alpha_i x_i]) =$$

$$= \mu \Sigma \alpha_i x_{i+1} \Rightarrow f([l_1, l_2]) \in \mathrm{alg}(f(l_1), f(l_2)) .$$

$$f(l_1 + l_2) = f(\mu b + \Sigma (\mu_i + \alpha_i) x_i) = \mu b + \Sigma (\mu_i + \alpha_i) x_i +$$

$$+ \Sigma (\mu_i + \alpha_i) x_{i+1} = f(l_1) + f(l_2) + [f(l_1), f(l_2)] \in \mathrm{alg}(f(l_1), f(l_2)).$$

Now suppose that

$$l_1 = \mu b + \Sigma \mu_i x_i \bar{\in} X \quad , \quad l_2 = \alpha b + \Sigma \alpha_i x_i \bar{\in} X$$

and suppose $\mu + \alpha \equiv 0 \pmod{p}$. There hold the following conditions:

$$f(l_1 + l_2) = f(\Sigma \mu_i x_i + \Sigma \alpha_i x_i) = \Sigma (\mu_i + \alpha_i) x_i ;$$

$$f(l_1) + f(l_2) = \mu b + \Sigma \mu_i (x_i + x_{i+1}) + \alpha b + \Sigma \alpha_i (x_i + x_{i+1}) =$$

$$= \Sigma (\alpha_i + \mu_i)(x_i + x_{i+1}) ;$$

$$f([l_1, l_2]) = f(\mu \Sigma \alpha_i x_{i+1} - \alpha \Sigma \mu_i x_{i+1}) = \mu \Sigma (\alpha_i + \mu_i)(x_i + x_{i+1}) ;$$

$$[f(l_1), f(l_2)] = [\mu b + \Sigma \mu_i (x_i + x_{i+1}) , \alpha b + \Sigma \alpha_i (x_i + x_{i+1})] =$$

$$= \alpha \Sigma \mu_i (x_{i+1} + x_{i+2}) - \mu \Sigma \alpha_i (x_{i+1} + x_{i+2}) .$$

On the other hand

$$\begin{array}{ccc}
\mathrm{alg}(\mu b + \Sigma \mu_i x_i , \alpha b + \Sigma \alpha_i x_i) & \xrightarrow{f} & \mathrm{alg}(\mu b + \Sigma \mu_i (x_i + x_{i+1}) , \alpha b + \Sigma \alpha_i (x_i + x_{i+1})) \\
\| & & \| \\
\mathrm{alg}(\mu b + \Sigma \mu_i x_i , \Sigma (\alpha_i + \mu_i) x_i) & \xrightarrow{f} & \mathrm{alg}(\mu b + \Sigma \mu_i (x_i + x_{i+1}) , \Sigma (\alpha_i + \mu_i) x_i) \\
\| & & \| \\
\mathrm{alg}(\alpha b + \Sigma \alpha_i x_i , \Sigma (\alpha_i + \mu_i) x_i) & \xrightarrow{f} & \mathrm{alg}(\alpha b + \Sigma \alpha_i (x_i + x_{i+1}) , \Sigma (\alpha_i + \mu_i) x_i) .
\end{array}$$

Consequently we have

$$\Sigma (\mu_i + \alpha_i) x_i , \Sigma (\mu_i + \alpha_i) x_{i+1} , \cdots \in \mathrm{alg}(f(l_1), f(l_2)) \Rightarrow$$

$$\Rightarrow f(l_1 + l_2), f([l_1, l_2]) \in \mathrm{alg}(f(l_1), f(l_2)) .$$

Suppose now that $\mu + \alpha \neq 0 \pmod{p}$, then there exists α_0 such that $\mu + \alpha_0 \alpha = 0 \pmod{p}$. It is clear that

$$f(l_1 + l_2) = f[(\mu + \alpha) b + \Sigma \mu_i x_i + \Sigma \alpha_i x_i = (\mu + \alpha) b +$$

$$+ \Sigma (\mu_i + \alpha_i) x_i + \Sigma (\mu_i + \alpha_i) x_{i+1} = f(l_1) + f(l_2) \in \mathrm{alg}(f(l_1), f(l_2)) .$$

Making use of the theorem of R. Baer [5], and the fact that on an one dimensional subalgebra from X f is induced by an isomorphism, because on an one dimensional subalgebras f is a projection, we have

$$\mathrm{alg}\,(f(l_1)\,,f(l_2)) = \mathrm{alg}\,(f(l_1)\,,\,\alpha_0 f(l_2)) \ni f([l_1\,,\,\alpha_0\, l_2]) =$$

$$= f(\alpha_0\,[l_1\,,\,l_2]) = \alpha_0 f([l_1\,,\,l_2]) \Rightarrow f([l_1\,,\,l_2]) \in \mathrm{alg}\,(f(l_1)\,,f(l_2))\,.$$

Consequently we can conclude that $f : \mathrm{L} \to \mathrm{L}$ induces an autoprojection and it is not induced by an isomorphism.

References

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