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On the classification of functions in separable metric spaces

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Matematica. — *On the classification of functions in separable metric spaces* (*). Nota di SANDRO LEVI, presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si definisce una classificazione delle funzioni su uno spazio metrico separabile basata sulle proprietà delle immagini degli insiemi aperti. Si stabiliscono inoltre alcuni risultati per le funzioni della prima classe e le funzioni aperte.

The classification of functions in metric spaces based on properties of the inverse images of open sets is well-known: as a general reference to the topic we quote the first volume of Kuratowski's monograph [1]. The purpose of this article is to assume a dual point of view and classify functions according to the properties of the images of open sets. Many results of the classical theory can be reformulated in the new setting but a basic difference with the previous situation appears: both the images of open and of closed sets have to be considered.

In this paper, however, the emphasis will be on the images of open sets. For a study of the Baire property along the same lines see [3].

We will define our classification, prove some general results, and then look at functions of the first class.

At the end of the paper we will define the analogue of the oscillation and discuss some of its applications.

X will denote a separable metric space, Y a topological space, α an ordinal smaller than ω_1-the first uncountable ordinal—and $f: X \to Y$ a function.

DEFINITION 1. Let Y be perfectly normal. The function f belongs to the direct class α—denoted by $f \in (\alpha)$—if the image of every open set under f is a Borel set of additive class α in Y.

We then have:

i) $f \in (0)$ iff f is open and $f \in (1)$ iff f maps open sets into F_σ sets

ii) every closed function belongs to (1) since each open set is an F_σ

iii) if $f \in (\alpha)$ and $\beta \geq \alpha$, $f \in (\beta)$

iv) if f maps open sets into Borel sets, there exists α such that $f \in (\alpha)$ since X has a countable open base

v) since Borel sets have the Baire property, if $f \in (\alpha)$ is onto, there exists a subset T of Y which is the complement of a set of the first category such that $f|_{f^{-1}(T)}: f^{-1}(T) \to T$ is open (see [3]).

(*) Lavoro eseguito nell'ambito del G.N.A.F.A. (C.N.R.).
(**) Nella seduta del 6 dicembre 1980.

THEOREM 2. *Let* Y *be perfectly normal.* $f \in (\alpha)$ *if and only if* $\forall \varepsilon > 0$ *there exists a sequence* $\{Z_n\}$ *of Borel subsets of* Y *of additive class* α *with the property that for every* $x \in X$ *there exist* $m \in N$ *and a subset* W_m *of* X *such that* $x \in W_m$, $f(W_m) = Z_m$ *and* $\operatorname{diam} W_m < \varepsilon$.

Proof. Let $f \in (\alpha)$. X being separable, there exists a sequence $\{B_n\}$ of open balls such that $X = \bigcup_n B_n$ and $\operatorname{diam} B_n < \varepsilon \ \forall n \in N$. Put $Z_n = f(B_n)$. Then the sequence $\{Z_n\}$ fulfills the requirements of the theorem.

Suppose conversely that the above-stated condition is verified. Then $\forall k \in N$ there is a sequence $\{Z_n^k\}_{n=1}^{\infty}$ such that each Z_n^k is of class α in Y. Let G be open in X and let $Z = \bigcup_{k,m} \{Z_m^k : \forall y \in Z_m^k \ f^{-1}(y) \cap G \neq \varnothing\}$. It is clear that $Z \subset f(G)$ and that Z is of additive class α.

Pick $y \in f(G)$ and $x \in G$ such that $y = f(x)$. Choose k and m so that the ball of radius $1/k$ centered at x is contained in G and there exists $W_m^k \subset X$ such that $x \in W_m^k$, $f(W_m^k) = Z_m^k$ and $\operatorname{diam} W_m^k < 1/k$. Then $W_m^k \subset G$, $y \in Z_m^k \subset Z$. Thus $Z = f(G)$ and the theorem is proved. //

As for sequences of functions of the direct class α, the usual concept of convergence is not suited to ensure that the limit function belong to the class (α).

Let $f_n : X \to Y$ be a sequence of surjective functions and let f be onto. We will impose the following conditions:

a) $\forall y \in Y \ f^{-1}(y) \subset \operatorname{Li} f_n^{-1}(y)$, that is $\forall x \in f^{-1}(y)$ there exist $x_n \to x$ with $x_n \in f_n^{-1}(y) \ \forall n \in N$.

b) $\forall \varepsilon > 0$, $\forall n \in N$ put $F_n^{\varepsilon} = \{z \in Y : \forall t \in f_n^{-1}(z) \ \exists x \in f^{-1}(z) : d(t, x) < \varepsilon\}$ where d is the distance in X. Then $\forall \varepsilon > 0$ every $y \in Y$ belongs to infinitely many F_n^{ε}.

THEOREM 3. *Let* Y *be perfectly normal and let* $\{f_n\}$ *and* f *verify conditions a) and b). Suppose further that each* $f_n \in (\alpha)$ *and that* $\forall \varepsilon > 0$, $\forall n \in N$, F_n^{ε} *is of additive class* α *in* Y. *Then* $f \in (\alpha)$.

Proof. Fix $\varepsilon > 0$ and put $F_n^{\varepsilon} = F_n$. Then $Y = \bigcup_n F_n$.

Since each $f_n \in (\alpha)$, there exists for every n a sequence $\{Z_i^n\}_{i=1}^{\infty}$ verifying the property of Theorem 2. Thus $Y = \bigcup_{n,i} (F_n \cap Z_i^n)$ where each $F_n \cap Z_i^n$ is of class α in Y.

Fix $\bar{x}$ in X. By *a)* and *b)* there exist $k \in N$ such that $f(\bar{x}) \in F_k$ and $t \in f_k^{-1}[f(\bar{x})]$ such that $d(\bar{x}, t) < \varepsilon$.

Given $k \in N$, $\exists m \in N$, $\exists W_n^k \subset X$ such that $t \in W_n^k$, $f_k(W_m^k) = Z_n^k$ and $\operatorname{diam} W_n^k < \varepsilon$. Furthermore for every $z \in F_k \cap Z_m^k$ and every $w \in f_k^{-1}(z) \cap W_m^k$ there exists $x \in f^{-1}(z)$ with $d(w, x) < \varepsilon$.

Put $W = \{x = x(w) : w \in f_k^{-1}(z) \cap W_m^k, \ z \in F_k \cap Z_m^k\} \cup \{\bar{x}\}$.

Then $f(W) = F_k \cap Z_m^k$ and $\operatorname{diam} W \leq 6\varepsilon$. Thus $f \in (\alpha)$. //

We are now going to consider the functions of the first direct class. Let us put $K = \{x \in X : f$ is not open at $x\}$ and let $\mathscr{B}$ be an open base for X.

LEMMA 4. $f(K) = \bigcup_{A \in \mathscr{B}} [f(A) - \operatorname{Int} f(A)]$.

Proof. If $x \in K$ there exists $A \in \mathscr{B}$ such that $x \in A$ and $f(x) \in f(A) - \operatorname{Int} f(A)$. Conversely let $y \in f(A) - \operatorname{Int} f(A)$ for some $A \in \mathscr{B}$. Then there is $x \in A$ such that $y = f(x) \cdot f(A)$ is not a neighbourhood of y and $x \in K$. //

THEOREM 5. *Suppose f maps open sets into F_σ sets. Then $f(K)$ is an F_σ of the first category in* Y.

Proof. Since X is separable it has a countable open base $\{G_n\}$.

For every n, $f(G_n) = \bigcup_{s=1}^{\infty} F_s^n$ where each F_s^n is closed.

Thus $f(G_n) - \operatorname{Int} f(G_n) = \bigcup_{s=1}^{\infty} F_s^n - \operatorname{Int} f(G_n)$ is an F_σ with empty interior and is therefore of the first category in Y.

By Lemma 4 the same conclusion holds for $f(K)$. //

COROLLARY 6. *Let* Y *be a Baire space and f a surjective function which maps open sets into F_σ sets. Then $Cf(K)$—the complement of $f(K)$—is a dense G_δ in* Y *and f is open at every $x \in f^{-1}[Cf(K)]$.*

As an immediate consequence we have Kuratowski's result ([2], p. 176) stated for X compact and f continuous.

COROLLARY 7. *Let* Y *be a Baire space and f a closed surjective function. Then there exists $B \subset X$ such that $f(\overline{B}) = Y$, f is open at every point of* B *and $f|_B : B \to f(B)$ is open.*

Proof. Put $B = f^{-1}[Cf(K)]$. Then $f(\overline{B}) = \overline{f(B)} = Y$ since f is closed and $f(B)$ is dense in Y. Let $x \in B$. By corollary 6 f is open at x: if A is any open neighbourhood of x, $f(A)$ is a neighbourhood of $f(x)$; since $f(A \cap B) = f(A) \cap f(B)$ it follows that $f|_B : B \to f(B)$ is open at x and thus open. //

The preceding corollary can be viewed as a partial dual (in the open-closed duality) of a result of Michael ([4] Corollary 1.2. (a)).

Let us now look at open function and define

$$\gamma_f(x) = \inf \{\operatorname{diam} Z : x \in Z \subset X, f(Z) \text{ is open in } Y\}$$

or $\gamma_f(x) = +\infty$ if there is no such Z.

LEMMA 8. *f is open at x if and only if $\gamma_f(x) = 0$.*

Proof. Let f be open at x and $\varepsilon > 0$. Put $S = S(x, \varepsilon/4)$, the open ball of radius $\varepsilon/4$ centered at x. Then $f(x) \in \operatorname{Int} f(S)$ and we can find $Z \subset S$ such that $x \in Z$ and $f(Z) = \operatorname{Int} f(S)$. Since $\operatorname{diam} S \leq \varepsilon/2$ we have $\gamma_f(x) < \varepsilon$ and we conclude that $\gamma_f(x) = 0$.

Suppose now that $\gamma_f(x) = 0$ and put $S_r = S(r, x)$ for $r > 0$.

There exists $Z \subset X$, $x \in Z$ such that $f(Z)$ is open in Y and diam $Z < r$. Then $Z \subset S_r$ and f is open at x. //

LEMMA 9. *If f is continuous at x_0, γ_f is upper semicontinuous at x_0.*

Proof. If $\gamma_f(x_0) = +\infty$ there is nothing to prove.

Let $\gamma_f(x_0) < r$ and choose $Z \subset X$ such that $x_0 \in Z$, $f(Z)$ is open in Y and diam $Z < r$. Since f is continuous at x_0, there exists a neighbourhood V of x_0 such that $V \subset f^{-1} f(Z)$.

Put $\varepsilon = r - \operatorname{diam} Z$ and $W = V \cap S(x_0, \varepsilon/2)$. Define $Z_x = Z \cup \{x\}$ $\forall x \in W$. Then $f(Z_x) = f(Z)$ and diam $Z_x \leq d(x, x_0) + \operatorname{diam} Z < r$.

We conclude that $\gamma_f(x) < r$ $\forall x \in W$ and γ_f is upper semicontinuous at x_0. //

Combining the two preceding lemmas we obtain the known.

COROLLARY 10. *If f is continuous the set of points at which f is open is a G_δ.*

Using the coefficient γ we can state a modified version of corollary 6 which is less precise as to the values of γ but is valid for open images.

THEOREM 11. *Let Y be a Baire space and f a surjective function which maps open sets into F_σ sets. Then $\forall n \in N$ there exists $A_n \subset X$ such that $f(A_n)$ is a dense open subset of Y and for every $x \in A_n$, $\gamma_f(x) < 1/n$.*

Proof. Let $\{x_m\}$ be a dense subset of X.

For every $n, m \in N$ let B^n_m be an open ball centered at x_m with diameter smaller than $1/n$.

Since $f \in (1) f(B^n_m) = \bigcup_{k=1}^{\infty} F^n_{m,k}$ where each $F^n_{m,k}$ is closed in Y. Y being a Baire space, $W_n = \bigcup_{k,m} \operatorname{Int} F^n_{m,k}$ is dense in Y.

If $y \in W_n$ there are m, k such that $y \in \operatorname{Int} F^n_{m,k}$ and $x \in B^n_m$ with $y = f(x)$. Since $\operatorname{Int} F^n_{m,k} \subset f(B^n_m)$ we see that $\gamma_f(x) < 1/n$.

To end the proof set

$$A_n = \{x : x \in B^n_m \cap f^{-1}(y) : y \in \operatorname{Int} F^n_{m,k} \text{ for some } m, k\}. \quad //$$

We conclude with two general remarks.

We can define, as was pointed out in the introduction, a new class $[\alpha]$ consisting of functions which map closed sets into Borel subsets of multiplicative class α. Then $[\alpha] \subset (\alpha + 1)$. But the classes $[\alpha]$ do not seem too handy due mainly to the fact that intersections and images do not commute.

Finally definition 1 can be stated in general metric spaces. It would be interesting to know to what extent the preceding results can be generalized.

References

[1] K. Kuratowski (1966) – *Topology*, vol. 1. New-York.

[2] K. Kuratowski (1926) – *Sur les décompositions semi-continues d'espaces metriques compacts*, « Fund. Math. », *11*, 169–185.

[3] S. Levi (1980) – *On the Baire property in separable metric spaces*. To appear in « Atti Accad. Naz. Lincei ».

[4] E. Michael (1959) – *A theorem on semi-continuous set-valued functions*, « Duke Math. J. », *26*, 647–651.