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On the Baire property in separable metric spaces

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On the baire property in separable metric spaces* (*). Nota di SANDRO LEVI, presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studiano le funzioni che applicano gli aperti in insiemi con la proprietà di Baire. Si caratterizzano gli spazi analitici che sono di Baire e si stabiliscono alcuni risultati sulla struttura degli spazi polacchi. Si dà infine una dimostrazione del teorema di Banach dell'applicazione aperta per gruppi separabili.

In this article we investigate the properties of functions defined on separable metric spaces which map open sets into sets that have the Baire property.

Much is known about functions for which the inverse images of open sets have the Baire property: see for instance [4].

We will first establish some preliminaries and give a characterization of analytic spaces which are Baire spaces, and then deduce, with the aid of a selection theorem due to Kuratowski and Maitra, some results on polish spaces; notably that every polish space is the completion of a suitable closed set of the irrationals for a metric which is equivalent to the euclidean metric.

As a last application we give a simple proof of the Banach open mapping theorem for groups.

(*) Lavoro eseguito nell'ambito del G.N.A.F.A. (C.N.R.).
(**) Nella seduta del 6 dicembre 1980.

X will denote a separable metric space, Y a topological space and $f: X \to Y$ a surjective function.

A subset B of X has the Baire property if it is open modulo first category sets, that is if $B = (A - P) \cup R$ where A is open in X and P and R are first category sets.

B has the Baire property in the restricted sense if $B \cap E$ has the Baire property in E for every E in X.

A set is analytic if it is the continuous image of a Borel subset of a complete separable space. Analytic sets have the Baire property in the restricted sense.

For these notions we refer to [4] (§ 11 and § 39).

DEFINITION 1. f has the direct Baire property (d.B.p.) if the image of every open set under f has the Baire property in Y.

THEOREM 2. *f has the* d.B.p. *if and only if there exists a set* P *of the first category in* Y *such that if we put* $g = f|_{f^{-1}(CP)}$, $g: f^{-1}(CP) \to CP$ *is open.* (CP *is the complement of* P).

Proof. Suppose f verifies the above-stated condition and let G be open in X. Then $G = [G \cap f^{-1}(CP)] \cup [G \cap f^{-1}(P)]$ and therefore

$$f(G) = [f(G) \cap CP] \cup [f(G) \cap P] = (W \cap CP) \cup [f(G) \cap P]$$

where W is open in Y. This proves that $f(G)$ has the Baire property in Y.

Conversely let f have the d.B.p. and let $\{G_n\}$ be an open base for X.

Then $\forall n \in N\, f(G_n) = (W_n - P_n) \cup R_n$ where W_n is open in Y and P_n and R_n are of the first category.

Put

$$P = \left(\bigcup_n P_n\right) \cup \left(\bigcup_n R_n\right).$$

Then P is of the first category.

Let H be open in $f^{-1}(CP)$. Then

$$H = \bigcup_{k=1}^{\infty} [G_{n_k} \cap f^{-1}(CP)]$$

and

$$g(H) = f(H) = \bigcup_k [f(G_{n_k}) \cap CP] = \bigcup_k \{[(W_{n_k} - P_{n_k}) \cup R_{n_k}] \cap CP\}$$
$$= \left(\bigcup_k W_{n_k}\right) \cap CP$$

and g is open. //

The next lemma furnishes a wide class of functions with the d.B.p.:

LEMMA 3. *Let* X *be a polish space,* Y *a metric space and* f *continuous.* Then f has the d.B.p.

Proof. Let $\tilde{Y}$ be the completion of Y. For every A open in X, $f(A)$ is analytic in $\tilde{Y}$ and therefore has the restricted Baire property. Thus $f(A) \cap Y = f(A)$ has the Baire property in Y and f has the d.B.p. //

The preceding Lemma and Theorem 2 enable us to characterize the analytic spaces that are Baire spaces:

THEOREM 4. *Let* X *be a polish space,* Y *a metric space and* f *continuous.* Then: i) Y is Baire if and only if there exists a dense G_δ D in Y such that $f|_{f^{-1}(D)} : f^{-1}(D) \to D$ is open;

ii) Y is of the second category (in itself) if and only if there exists a G_δ D′ dense in some non-empty open subset of Y such that $f|_{f^{-1}(D')} : f^{-1}(D') \to D'$ is open.

Proof. *i*) Suppose Y is Baire. By Lemma 3 and Theorem 2 there exists a first category set P in Y such that $f|_{f^{-1}(CP)}$ is open. P is contained in a first category F_σ, P′. Then $D = CP'$ is a dense G_δ and $f|_{f^{-1}(D)}$ is open.

Suppose conversely that the condition of the theorem is verified.

Then $f^{-1}(D)$ is a G_δ in X and thus topologically complete. By Hausdorff's theorem ([2] th. 1) D is also topologically complete and hence a Baire space. It follows that Y is a Baire space.

ii) follows from *i*) and the fact that a topological space is of the second category if and only if it contains a non-empty open Baire subspace ([3] th. 1.26). //

COROLLARY 5. *Suppose* X *is a separable metrizable space under two topologies* τ_1 *and* τ_2 *and that* τ_1 *is finer than* τ_2. *Suppose further that* (X, τ_2) *is Baire and that every* τ_1*-open set has the Baire property for* τ_2 *(the last condition is verified if* (X, τ_1) *is topologically complete). Then there is a subset* D *which is a dense* G_δ *for* τ_2 *on which* τ_1 *and* τ_2 *agree.*

As an example we can consider the closed unit ball of a separable Hilbert space under the norm and weak topologies.

We now derive some results on the structure of polish spaces.

THEOREM 6. *Let* X *be a polish space,* Y *a Baire metric space and* f *continuous. Then* f *induces by restriction a homeomorphism between a* G_δ *in* X *and a dense* G_δ *in* Y.

Proof. By Theorem 4 *i*) there exists a dense G_δ D in Y such that

$$f_D = f|_{f^{-1}(D)} : f^{-1}(D) \to D \qquad \text{is open.}$$

By a result of Kuratowski and Maitra [5] there is a G_δ selection S in $f^{-1}(D)$ for the partition induced by f_D.

Put $g = f_D|_S$; then $g : S \to D$ is continuous and one-to-one.

Again by Theorem 4 there exists a dense G_δ G in D such that

$$h = g|_{g^{-1}(G)} : g^{-1}(G) \to G \text{ is open.}$$

h is a homeomorphism and G is dense in Y. //

COROLLARY 7. *If* X *and* Y *are polish spaces and* f *is continuous,* Y *is homeomorphic to the completion* $(H, \rho)^{\sim}$ *where* H *is a* G_δ *in* X *and* ρ *a metric compatible with the topology of* H.

Proof. Let d be a metric on Y such that (Y, d) is complete. With h and G as in the preceding theorem put $H = h^{-1}(G)$ and define

$$\rho(x, x') = d[h(x), h(x')] \; \forall x, x' \in H.$$

Then $(H, \rho)^{\sim} = (G, d)^{\sim} = (Y, d)$, where the equal sign means "isometric to". //

COROLLARY 8. *Let* X *be any polish space. Then there exist a closed subset* H *of the irrationals and a metric* ρ *on* H *which is equivalent to the euclidean metric such that* X *is homeomorphic to* $(H, \rho)^{\sim}$.

Proof. Given X there is a continuous map of the irrationals onto X. Since every G_δ of the irrationals is homeomorphic to a closed set ([4] § 36 II), it is enough to apply corollary 7 to complete the proof. //

COROLLARY 9. *Every polish space admits a dense* G_δ *of dimension zero.*

We will now give an application to topological groups. (Cf. Banach [1] chapters I and III).

We will use the following known:

LEMMA 10. *Let* T *and* V *be topological spaces verifying the first axiom of countability and* $g: T \to V$ *surjective. Then* g *is open at* $x \in T$ *if and only if for every sequence* $y_n \to g(x)$, *there exists a sequence* $x_n \to x$ *such that* $y_n = g(x_n) \; \forall n \in N$.

A simple proof of this lemma will be given at the end of the paper.

THEOREM 11. *Let* X *and* Y *be topological groups with* Y *a Baire space satisfying the first axiom of countability. Let* f *be a surjective homomorphism with the* d.B.p. *Then* f *is open.*

Proof. It is enough to prove that f is open at e, the neutral element of X. Let $y_n \to e'$ (the neutral element of Y).

By Theorem 2 there exists P of the first category in Y such that $g = f|_{f^{-1}(CP)}$ is open.

$y_n P$ is of the first category for every n and so is $P \cup \left(\bigcup_n y_n P\right)$.

Since $C(y_n P) = y_n CP$, $CP \bigcap_n \left(\bigcap_n y_n CP\right)$ is non-empty and let y belong to this subset. Thus y and $y_n^{-1} y \in CP \quad \forall n \in N$.

Let $x \in f^{-1}(y)$. Since g is open at x and $y_n^{-1} y \to y$, by Lemma 10 there is a sequence $x_n \to x$ such that $f(x_n) = y_n^{-1} y$ $\forall n \in \mathrm{N}$.

But then $x x_n^{-1} \to e$ and $f(x x_n^{-1}) = f(x) f(x_n)^{-1} = y y^{-1} y_n = y_n$.

This shows that f is open at e. //

COROLLARY 12. *Let* X *and* Y *be topological groups with* X *a polish space and* Y *Baire metric. Let* f *be a continuous surjective homomorphism. Then* f *is open.*

The proof follows from Lemma 3 and the preceding theorem.

We conclude by proving Lemma 10 in the case T is a metric space. In [6] we define the coefficient γ as

$$\gamma_g(x) = \inf \{\operatorname{diam} Z : x \in Z , g(Z) \text{ is open in } V\}$$

or $\gamma_g(x) = +\infty$ if there is no such Z, and proved that g is open at x if and only if $\gamma_g(x) = 0$.

Suppose now that g is not open at x. Then there exists a neighbourhood A of x such that $y = g(x) \notin \operatorname{Int} g(A)$.

Let $\{E_n\}$ be a decreasing base of neighbourhoods of y. Then $\forall n \in \mathrm{N}$ there is $y_n \in E_n - g(A)$ and the sequence $\{y_n\}$ tends to y. Suppose that $x_n \to x$ and that $g(x_n) = y_n$ for every n. There exists $n_0 : \forall n > n_0$ $x_n \in A$ which implies $y_n \in g(A)$. Thus no such sequence $\{x_n\}$ can exist.

Suppose that g is open at x. Then $\gamma_g(x) = 0$ and for every $k \in \mathrm{N}$ there is Z_k such that $x \in Z_k$, $g(Z_k)$ is open and $\operatorname{diam} Z_k < 1/k$.

We can suppose that the $g(Z_k)$'s are decreasing. Let $y_n \to y$.

Take $n = 1$ and let $n_1 \in \mathrm{N}$ be the smallest index such that $y_n \in g(Z_1)$ for every $n \geq n_1$. Select $x_{n_1} \in Z_1$ such that $g(x_{n_1}) = y_{n_1}$.

Suppose we have chosen x_{n_k}. Let n_{k+1} be the smallest index greater than n_k and such that $y_n \in g(Z_{k+1})$ for every $n \geq n_{k+1}$.

Select $x_{n_{k+1}} \in Z_{k+1}$ such that $g(x_{n_{k+1}}) = y_{n_{k+1}}$.

It is clear that $\lim_{k \to \infty} x_{n_k} = x$.

We complete the sequence $\{x_n\}$ by picking any $x_n \in g^{-1}(y_n)$ for every $y_n \notin g(Z_1)$ and by selecting any $x_n \in Z_k \cap g^{-1}(y_n)$ for every $n_k < n < n_{k+1}$. //

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