Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the existence of normal Sylow p-complements

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **69** (1980), n.5, p. 228–231.

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Algebra. — *On the existence of normal Sylow p-complements.*
Nota (*) di ANNA LUISA GILOTTI e LUIGI SERENA, presentata (**) dal Socio G. ZAPPA.

RIASSUNTO. — Nella Nota gli autori danno una nuova condizione necessaria e sufficiente per l'esistenza di p-complementi di Sylow normali nei gruppi finiti.

INTRODUCTION

In this note we give a new condition on the existence of normal p-complements in finite groups. Such a condition uses the notion of p-W-chain i.e. a chain of the following type:

$$\langle 1 \rangle = V_0 \trianglelefteq V_1 \trianglelefteq \cdots \trianglelefteq V_r = S$$

where S is a Sylow p-subgroup of G, V_i $(i = 1, \cdots, r)$ is weakly closed subgroup in S w.r. to G, $V_i/V_{i-1} \leq Z(S/V_{i-1})$.
This definition generalizes the p-I-chain definition, introduced in [2].

The main result is the following

THEOREM A. *A finite group* G *has a normal p-complement if and only if the following conditions are satisfied*:

1) G *has a* p-W-*chain*
2) $(|N_G(S) : C_G(S)|, p^h - 1) = 1$ *for all* h, $1 \leq h \leq n$

where $n = \max r_i$, $r_i =$ *number of generators of* V_i/V_{i-1}. $(i = 1, \cdots, r)$

The existence of the kernel in a Frobenius group where the complement is a p-group is an example of immediate consequence of Theorem A.

In the second section the theorem is used to give conditions for the existence of non trivial p-factors.

Any group is finite, the notation is standard (cf. [3]). In particular, $S_p(G)$ is the set of Sylow p-subgroups of G and S indicates an element of $S_p(G)$.

1. First of all we prove the elementary

LEMMA. *Let* N *be a normal subgroup of* G, V *be a subgroup which is weakly closed in* S *w.r. to* G. *Then* VN/N *is weakly closed in* SN/N *w.r. to* G/N.

(*) Eseguita nell'ambito dell'attività del G.N.S.A.G.A. del. C.N.R.
(**) Nella seduta dell'8 novembre 1980.

Proof. Let VN/N , V^xN/N be contained in the same Sylow p-subgroup SN/N of G/N. S $\in S_p$(SN). Since V^x is a p-subgroup of SN, there exists $y \in$ N such that $V^x \leq S^y$. Since V is weakly closed in S w.r. to G, and $V^x, V^y \leq S^y$, then $V^x = V^y$. It follows $V^x N = V^y N$. Since $y \in N$, $V^y N = VN$. So $V^x N/N = VN/N$.

Proof of Theorem A.

Necessity. Let $N = N_G(S)$. Since G has a normal p-complement K, $K \cap N$ is a normal p-complement in N. Then $N = (K \cap N) \times S$. So $|N_G(S) : C_G(S)|$ is a power of p. By Theorem 2.2 [2] G has a p-I-chain, thus, since $n = 1$, $(|N_G(S) : C_G(S)|, p - 1) = 1$.

Sufficiency. Assume that G verifies the hypotheses 1 and 2 of Theorem A. Let $\langle 1 \rangle = V_0 \trianglelefteq V_1 \trianglelefteq \cdots \trianglelefteq V_r = S$ be a p-W-chain of G. We prove the theorem by induction on $|G|$. Let $N_1 = N_G(V_1)$. If $N_1 < G$, since N_1 verifies the same hypotheses as G, N_1 has a normal p-complement. V_1 is weakly closed in S and $V_1 \leq Z(S)$, then by Grun's II theorem and Tate's theorem, $N_1/O^p(N_1) \simeq G/O^p(G)$.
Hence G has a normal p-complement and we are done.
So we can assume $N_1 = G$. Let $\overline{G} = G/V_1$, and $\overline{V}_i = V_i/V_1$ $(i = 2, \cdots, r)$.
By Lemma, $\overline{V}_i$ $(i = 2, \cdots, r)$ is weakly closed in $\overline{S} = \overline{V}_r$ w.r. to G.
Furthermore, $\overline{V}_i/\overline{V}_{i-1} \leq Z(\overline{S}/\overline{V}_{i-1})$ and $\overline{V}_i/\overline{V}_{i-1} \cong V_i/V_{i-1}$.
Also $N_{\overline{G}}(\overline{S}) = N_G(S)/V_1$ and $C_{\overline{G}}(\overline{S}) \geq C_G(S)/V_1$, therefore $(|N_{\overline{G}}(\overline{S}) : G_{\overline{G}}(\overline{S})|, p^h - 1) = 1$ for all h, $1 \leq h \leq n$, $n =$ number of generators of $\overline{V}_i/\overline{V}_{i-1}$ $(i = 2, \cdots, r)$. By induction, G has a normal p-complement $\overline{K} = K/V_1$. If K has a normal p-complement, the same is true for G. So let us assume that K has not a normal p-complement. Let $O_{p'}(G) \neq 1$, and N be a normal p'-subgroup of G, $N \neq 1$. Let $G^* = G/N$ and A^* be the image of a subgroup A of G in the natural homomorphism of G over G^*. By Lemma $\langle 1 \rangle = V_0^* \leq V_1^* \leq \cdots \leq V_r^* = S^*$ is a p-W-chain of G^*. Also $V_i^*/V_{i-1}^* \cong V_i/V_{i-1}$ $(i = 1, \cdots, r)$ and $|N_{G^*}(S^*) : C_{G^*}(S^*)|$ divides $|N_G(S) : C_G(S)|$, so G^* verifies the same hypotheses as G. By induction, G^* has a normal p-complement R^* whose preimage R is a normal p-complement of G. So we can assume $O_{p'}(G) = 1$.

Since V_1 is a normal Sylow p-subgroup of K, by the Schur-Zassenhaus theorem, K has a p-complement M and all the complements are conjugate. By Frattini's argument $G = KN_G(M) = V_1 N_G(M)$.

Let us consider the action of G on the set Ω of the conjugates of M. G acts transitively on Ω, so if Q is the kernel of this representation, $\hat{G} = G/Q$ is a transitive permutation group on Ω. We want to prove that Q is a p-group.

Let a be a p'-element of Q. Since $Q = \bigcap_{x \in G} N_G(M^x)$, $\langle M^x, a \rangle$ is a p'-group for any $x \in G$. But M^x is a Hall p'-subgroup, then $a \in M^x$. It follows that $a \in \bigcap_{x \in G} M^x \leq O_{p'}(G) = \langle 1 \rangle$.

Thus Q is a p-group and, since $Q \trianglelefteq G$, $Q \leq S$.
Since $G = V_1 N_G(M)$, V_1 acts transitively on Ω. So $\hat{V}_1 = V_1 Q/Q$ is a transitive subgroup of $\hat{G}$. V_1 is abelian, so it is self-centralizing in $\hat{G}$. Let $\hat{S} = S/Q$. From $V_1 \leq Z(S)$ it follows $\hat{V}_1 \leq Z(\hat{S})$, and then $\hat{V}_1 = C_{\hat{G}}(\hat{V}_1) \geq \hat{S}$. Therefore $\hat{V}_1 = \hat{S}$, i.e. $S = V_1 Q$. So S is normal in G. $G = N_G(S)$, $C_G(S) \leq C_G(V_1)$. It follows that $| N_G(V_1) : C_G(V_1) |$ divides $| N_G(S) : C_G(S) |$. So $| N_K(V_1) : C_K(V_1) |$ is prime to $| \mathrm{Aut}(V_1) |$ by theorem 3.19 (pag. 275, [7]). It follows $N_K(V_1) = C_K(V_1)$. By Burnside's theorem, K has a normal p-complement, a contradiction.

COROLLARY. *A finite group* G *has a normal Sylow p-complement if and only if* G *has a chain of the type*:

$$\langle 1 \rangle = V_0 \trianglelefteq V_1 \trianglelefteq \cdots \trianglelefteq V_r = S$$

where V_i $(i = 1, \cdots, r)$ *is weakly closed in* S *w.r. to* G, $V_i/V_{i-1} \leq Z(S/V_{i-1})$, V_i/V_{i-1} *cyclic and* $(| N_G(S) : C_G(S) |, p - 1) = 1$.

2. DEFINITION. *Let* V *be a subgroup which is weakly closed in* S *w.r. to* G. *We will call p*-W-*chain connecting* V *and* S *any chain of the type*:

$$V = V_0 \trianglelefteq V_1 \trianglelefteq \cdots \trianglelefteq V_r = S$$

where V_i $(i = 1, \cdots, r)$ *is weakly closed in* S *w.r. to* G *and* $V_i/V_{i-1} \leq Z(S/V_{i-1})$.

PROPOSITION B. *Let* V *be a subgroup which is weakly closed in* S *w.r. to* G, $V \neq S$. *Suppose that* $[x, \underbrace{y, \cdots, y}_{p-1}] \in \Phi^*(V)$, $x \in S$, $y \in V$ (*where* $\Phi^*(V)$ *is the intersection of the subgroups of* V *of index at most* p^2). *Furthermore suppose that*

1) G *has a p*-W-*chain connecting* V *and* S

2) $(| N_G(S) : C_G(S) |, p^h - 1) = 1$ *for all* h, $1 \leq h \leq n$

where $n = max\ r_i$, r_i = *number of generators of* V_i/V_{i-1} $(i = 1, \cdots, r)$.
Then $O^p(G) < G$.

Proof. By induction on $|G|$. Let $N = N_G(V)$. If $N < G$, N verifies the same hypotheses as G, thus, by induction $O^p(N) < N$. By Corollary 4.3.1 ([5]), $N/O^p(N) \cong G/O^p(G)$. So $O^p(G) < G$.

Let us assume $N = G$. Let $\bar{G} = G/V$. If $V = 1$, by Theorem A we have the desired conclusion. If $V \neq 1$, $|\bar{G}| < |G|$, and $\bar{G}$ has a p-W-chain. Let $\bar{S} = S/V$. We have $N_{\bar{G}}(\bar{S}) = N_G(S)/V$ thus $| N_{\bar{G}}(\bar{S}) : C_{\bar{G}}(\bar{S}) |$ divides $| N_G(S) : C_G(S) |$. Then by induction, $O^p(\bar{G}) < \bar{G}$ and so $O^p(G) < G$.

Remark. By using the same proof and the Corollary 4.6.2 [5] we can prove the following

PROPOSITION C. *Let* V *be a subgroup which is strongly closed in* S *w.r. to* G, *and weakly regular (for the definition see* [5]).

Let $V \neq S$. *Suppose that* G *satisfies the hypotheses* 1 *and* 2 *of Proposition B.*

Then $O^p(G) < G$.

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