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On the existence of scrolls in P^4

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Geometria. — *On the existence of scrolls in* $\mathbf{P}^4$ (*). Nota di ANTONIO LANTERI (**), presentata (***) dal Socio G. ZAPPA.

RIASSUNTO. — Si dimostra il seguente risultato. Sia X una superficie proiettivamente rigata, non iperpiana, di $\mathbf{P}^4$; allora X è la rigata cubica oppure è una rigata quintica ellittica. Si descrive inoltre una nuova generazione proiettiva delle rigate quintiche ellittiche di $\mathbf{P}^4$.

1. INTRODUCTION

In this paper we prove the following: let $X \subset \mathbf{P}^4$ be a scrollar surface not lying in any hyperplane (we consider only smooth surfaces); then X is either the cubic rational scroll or the quintic scroll over an elliptic curve. This confirms a conjecture expressed in a previous paper on surfaces in $\mathbf{P}^4$ ([4]), where the same fact was proved only for scrollar surfaces of degree $d < 11$. In particular it turns out that in $\mathbf{P}^4$ there are no scrollar surfaces with irregularity $q > 1$ and this agrees with a circulating conjecture on the existence of a bound for the irregularity of the surfaces in $\mathbf{P}^4$. In fact, but the elliptic scrolls, a unique class of irregular surfaces in $\mathbf{P}^4$ is still known: the abelian surfaces of degree $d = 10$ studied by Horroks and Mumford ([2]).

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Here is a sketch of our proof. Consider a scroll $X \subset \mathbf{P}^4$ of degree d and the associate curve C_X in the Grassmannian of the lines of $\mathbf{P}^4$. By means of a basic formula proved in [4] we can express the genus of C_X as a polynomial in d. Then by applying to C_X the Castelnuovo inequality for the genus of a curve in $\mathbf{P}^n$ we prove that $d \leq 5$. This is enough to conclude.

In the last section we supply a new projective construction of the quintic scroll in $\mathbf{P}^4$ over a given elliptic curve.

2. $\mathbf{P}^4$ CONTAINS NO SCROLL OF DEGREE $d > 5$

Let $X \subset \mathbf{P}^n$ be a complex irreducible smooth algebraic surface; if there exists a morphism $p : X \to B$ over an (irreducible and smooth) curve B, each fibre of which is a line, X is said to be a *scroll* over B. Denote by $q(X)$ and $g(B)$ the irregularity of X and the genus of B respectively. If X is a scroll over B, then $q(X) = g(B)$.

Throughout this paper we consider scrolls embedded in the four dimensional projective space $\mathbf{P}^4$. First of all we have the following basic formula.

LEMMA 2.1. *Let* $X \subset \mathbf{P}^4$ *be a scroll of degree d and irregularity q. Then*

$$q = \frac{1}{6}(d-2)(d-3). \tag{2.1}$$

For a proof see [4], Proposition 3.1.

Consider now the Grassmann manifold $G = \mathrm{Grass}(2, 5)$ of the lines of $\mathbf{P}^4$. As it is well known, G is a six dimensional algebraic manifold of degree five embedded in $\mathbf{P}^9$. Denote by C_X the curve in G corresponding to a scroll $X \subset \mathbf{P}^4$, and by $\langle C_X \rangle$ its linear span.

Remark 2.1. Let $X \subset \mathbf{P}^4$ be a scroll of degree d and irregularity q. Then

i) C_X is a smooth curve of degree d and genus q;

ii) if $d > 5$, $\dim \langle C_X \rangle \geq 5$.

For *i*) see [5], p. 281. To see *ii*) suppose $\dim \langle C_X \rangle \leq 4$. Then C_X is a component of the curve section of G with a four dimensional linear space L of $\mathbf{P}^9$. As G has degree five, this implies $d \leq 5$.

Consider now integers r, h, k, such that

$$0 \leq k \leq r - 1, \tag{2.2}$$

and the polynomial

$$F(r, h, k) = r(r-3)h^2 + 2k(r-3)h + (k-1)(k-2). \tag{2.3}$$

LEMMA 2.2. *Suppose*

$$4 \leq r \leq 8 , \tag{2.4}$$

and $h \geq 2$; *then*

$$F(r, h, k) > 0 . \tag{2.5}$$

Proof. Consider $F_{r,k}(h) = F(r, h, k) \in \mathbf{Z}[r, k][h]$, and denote by $h_1 = h_1(r, k), h_2 = h_2(r, k)$ $(h_1 \leq h_2)$ the roots of the equation $F_{r,k}(h) = 0$. Suppose $F(r, h, k) \leq 0$; it is sufficient to see that if (2.4) holds, then

$$h_2 < 2 . \tag{2.6}$$

Let $\Delta = \Delta(r, k)$ be the discriminant of $F_{r,k}(h)$ and $k_m = \frac{1}{2} r$. Then $\Delta(r, k_m) \geq \Delta(r, k)$ if r and k satisfy (2.4) and (2.2) respectively. Now, recalling (2.3), we have

$$h_2 \leq \frac{1}{r}(\sqrt{\Delta/4} - k) \leq \frac{1}{r}\sqrt{\Delta/4} \leq \frac{1}{r}\sqrt{\Delta(r, k_m)/4} .$$

Thus a straightforward calculation of $\Delta(r, k_m)$ with r as in (2.4) gives (2.6).

It is well known that $\mathbf{P}^4$ contains a cubic rational scroll (i.e. the Steiner surface of $\mathbf{P}^4$) and the quintic elliptic scrolls with invariant $e = -1$ corresponding to the general curve sections of the Grassmannian G. It is also known that $\mathbf{P}^4$ contains no other scroll of degree $3 \leq d \leq 5$. Now we can state the following

THEOREM 2.1. *Let* $X \subset \mathbf{P}^4$ *be a scroll (not lying in any hyperplane* (1)); *then either*

i) X *is the cubic scroll, or*

ii) X *is a quintic elliptic scroll.*

Proof. If X has degree $d \leq 5$, the theorem is trivial. Suppose X is a scroll of degree $d > 5$ and consider the curve C_X. By Remark 2.1 we can suppose $\dim \langle C_X \rangle = r + 1$ where r satisfies (2.4). Therefore, since C_X is a curve in $\mathbf{P}^{r+1}$ not lying in any hyperplane, its genus $g(C_X)$ must satisfy the following inequalities (cf. [1], p. 253):

$$g(C_X) \leq \begin{cases} d - r - 1 & \text{if} \quad r + 1 < d \leq 2r + 1 , \\ r + 2 & \text{if} \quad d = 2r + 2 , \\ \binom{m}{2} r + m\varepsilon & \text{if} \quad d > 2r + 2 , \end{cases} \tag{2.7}$$

(1) The unique scroll in $\mathbf{P}^3$ is the quadric surface.

where $m = [(d-1)/r]$, $\varepsilon = d - 1 - mr$ and $[\ ]$ is the least integer function. On the other hand, $g(C_X) = q(X)$ and by Lemma 2.1, q is given by (2.1). Now, if $r+1 < d \leq 2r+2$ it is easy to see that the first two inequalities in (2.7) cannot be satisfied. Otherwise, if $d > 2r+2$ we can write $d - 1 = hr + k$ where

$$(2.8) \qquad h \geq 2 \quad \text{and} \quad \begin{matrix} 0 \\ 2 \end{matrix} \Big\} \leq k \leq r-1 \quad \begin{matrix} \text{if } h > 2 \\ \text{if } h = 2 ; \end{matrix}$$

thus $m = h$ and $\varepsilon = k$ in (2.7) and the third inequality in (2.7) is equivalent to

$$F(r, h, k) \leq 0 .$$

But, in view of (2.4) and (2.8), this inequality contradicts Lemma 2.2.

3. A projective generation of the quintic elliptic scroll

Let B be an elliptic curve and denote by X(B) the quintic elliptic scroll over B contained in $\mathbf{P}^4$. Several ways to give an explicit construction of X(B) are known. For instance, X(B) can be generated by intersecting five suitable linear complexes of $\mathbf{P}^4$ (cf. [5], p. 278) or by means of two elliptic cubic curves isomorphic to B meeting in a single point (cf. [3], p. 232). In this sec. we supply a different construction of X(B) which seems to be new. The key of this construction is the existence of elliptic two-secant curves on the $\mathbf{P}^1$-bundle of invariant $e = -1$ over an elliptic curve (2).

Suppose C is a smooth elliptic curve of degree $d = 5$ in $\mathbf{P}^4$ not contained in any hyperplane. Consider a nontrivial fixed-point free involution σ of C (i.e. a translation of half a period if we think of C as a complex torus), the elliptic curve $B = C/(\sigma)$, the projection $\pi : C \to B$ and the line

$$(3.1) \qquad F_b = \langle p, \sigma(p) \rangle \qquad (p \in C \text{ and } b = \pi(p) = \pi(\sigma(p))) .$$

LEMMA 3.1. *If $b, b' \in B$, $b \neq b'$, the lines F_b and $F_{b'}$ are skew.*

Proof. Suppose $F_b \cap F_{b'} \neq \varnothing$. Then $\langle F_b, F_{b'} \rangle$ is a plane. Consider the pencil $\{\Pi_t\}_{t \in \mathbf{P}^1}$ of hyperplanes through $\langle F_b, F_{b'} \rangle$. As $\deg C = 5$, a nonconstant morphism $\mathbf{P}^1 \to C$ is defined which associates to Π_t the point which it cuts on C outside the four base points. This is absurd since $g(C) = 1$.

Thus we deduce

PROPOSITION 3.1. *The surface S generated by the lines F_b $(b \in B)$ is the quintic elliptic scroll* X(B).

(2) Really this bundle admits three two-secant curves (cf. [6], p. 310).

Proof. In fact S is a scroll over B by Lemma 3.1, and formula (2.1) gives $d = 5$ (3).

Now to construct X(B) for a given B consider: a double unramified covering $\overline{\pi} : \overline{C} \to B$, a quintic elliptic smooth curve $C \subset \mathbf{P}^4$ isomorphic to $\overline{C}$, the involution σ corresponding to $\overline{\pi}$ and define the line F_b as in (3.1). Proposition 3.1 tells us that X(B) is the surface generated by the lines F_b ($b \in B$).

REFERENCES

[1] P. GRIFFITHS and J. HARRIS (1978) – *Principles of Algebraic Geometry*. John Wiley and Sons, Inc. New York.

[2] G. HORROKS and D. MUMFORD (1973) – *A rank 2 vector bundle on* $\mathbf{P}^4$ *with 15,000 symmetries*, «Topology», *12*, 63–81.

[3] A. LANTERI and M. PALLESCHI (1978) – *Osservazioni sulla rigata geometrica ellittica di* $\mathbf{P}^4$ «Istituto Lombardo (Rend. Sc.)», *A 112*, 223–233.

[4] A. LANTERI and M. PALLESCHI (1979) – *Sulle superfici di grado piccolo in* $\mathbf{P}^4$. «Istituto Lombardo (Rend. Sc.)», *A 113*, 224–241.

[5] J. G. SEMPLE and L. ROTH (1949) – *Introduction to Algebraic Geometry*. Clarendon Press, Oxford.

[6] T. SUWA (1969) – *On ruled surfaces of genus 1*, «J. Math. Soc. Japan», 21, 291–311.

(3) It can also be directly seen that S has degree $d = 5$. In fact d is the number of lines F_b which intersect a general plane L in $\mathbf{P}^4$. The map σ allows us to define a correspondence ψ of bidegree [5, 5] in the pencil of hyperplanes through L. As σ is an involution the ten united points of ψ correspond to five hyperplanes through L containing a pair $(p, \sigma(p))$; obviously each pair defines a line F_b intersecting L.