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θ-Runge domains for P-holomorphy types

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *θ-Runge domains for P-holomorphy types.* Nota di LUIZA A. MORAES, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Vengono studiati i legami fra i domini di Runge e domini di Runge di tipo θ in uno spazio di Banach complesso, introducendo una nuova nozione di dominio di Runge.

1. θ-RUNGE DOMAINS IN BANACH SPACES WITH THE STRONG APPROXIMATION PROPERTY

Let θ be a holomorphy type from E to $\mathbf{C}\mathcal{P}_\theta(\mathrm{E})$ will denote the set of all $p = \sum_{i=0}^{n} p_i$ where

$$p_i \in \mathcal{P}_\theta(^i\mathrm{E}) \qquad \forall i = 0, \cdots, n; \qquad n \in \mathbf{N}.$$

DEFINITION 1.1. A Hausdorff locally convex space E has the strong approximation property (SAP) if there exists a family $\mathcal{F}_\mathrm{E}$ of continuous projections with finite range such that $(u(\mathrm{E}))_{u \in \mathcal{F}}$ satisfies the following condition: for every compact subset K of E and for every neighbourhood V of zero, there exists $u \in \mathcal{F}_\mathrm{E}$ such that $u(x) - x \in \mathrm{V}$ for all $x \in \mathrm{K}$.

DEFINITION 1.2. We say that θ is a P-holomorphy type if given any continuous projection with finite dimensional range $u : \mathrm{E} \to \mathrm{E}$ we have $p_i \circ u \in \mathcal{P}_\theta(^i\mathrm{E})$ for all $p_i \in \mathcal{P}(^i\mathrm{E}_u)$ for any $i \in \mathbf{N}$. $(\mathrm{E}_u = u(\mathrm{E}))$.

(*) Nella seduta del 12 aprile 1980.

Some examples of P-holomorphy type are:

1) The current type
2) The compact type
3) The nuclear type.

PROPOSITION 1.1. *The following statements are equivalent*:

(1) A is a P-holomorphy type
(2) $\mathcal{P}_\theta({}^n\mathrm{E}) \supset \mathcal{P}_f({}^n\mathrm{E}) \quad \forall n \in \mathbf{N}$
(3) For all continuous projections $u : \mathrm{E} \to \mathrm{E}$ such that $\dim u(\mathrm{E}) = 1$ and for all $\mathrm{P} \in \mathcal{P}({}^n\mathrm{E}_u)$ we have $\mathrm{P} \circ u \in \mathcal{P}_\theta({}^n\mathrm{E})$.

Notation. $\hat{\mathrm{K}}_{\mathcal{P}_\theta}(\mathrm{E}) = \{\xi \in \mathrm{E} : |f(\xi)| \leq \|f\|_\mathrm{K} \ \forall f \in \mathcal{P}_\theta(\mathrm{E})\}$

$\mathrm{K} \subset \mathrm{U} : \hat{\mathrm{K}}_\theta = \{\xi \in \mathrm{E} : |f(\xi)| \leq \|f\|_\mathrm{K} \ \forall f \in \mathcal{H}_\theta(\mathrm{U})\}$.

DEFINITION 1.3. An open subset U of E is $\mathcal{P}_\theta(\mathrm{E})$-convex if $\hat{\mathrm{K}}_{\mathcal{P}_\theta} \cap \mathrm{U}$ is compact for every compact set K contained in U. It is θ-holomorphically convex if $\hat{\mathrm{K}}_\theta$ is compact for every compact set $\mathrm{K} \subset \mathrm{U}$.

DEFINITION 1.4. An open subset U of a Banach space E is a θ-Runge domain if $\mathcal{P}_\theta(\mathrm{E})$ is dense in $(\mathcal{H}_\theta(\mathrm{U}), \mathcal{T}_0)$.

PROPOSITION 1.2. *Let* E *be a Banach space,* θ *a* P-*holomorphy type from* E *to* **C** *and* U *a* θ-*holomorphically convex open subset of* E. *Then if* U *is a* θ-*Runge domain,* U ∩ F *is a Runge domain for each finite dimensional subspace* F *of* E.

THEOREM 1.1. *Let* E *be a Banach space with the* SAP, θ *a* P-*holomorphy type from* E *to* **C** *and* U *a* θ-*holomorphically convex open subset of* E. *Then* U *is a* θ-*Runge domain iff* U *is a Runge domain.*

THEOREM 1.1′. *Let* E *be a Banach space with the* SAP *and* U *be an open subset of* E. *Let* θ *be a* P-*holomorphy type from* E *to* **C**. *Then* U *is a* θ-*Runge domain if* U *is a Runge domain.*

PROPOSITION 1.3. *Let* E *be a Banach space with the* SAP, θ *a* P-*holomorphy type from* E *to* **C** *and* U *a* θ-*holomorphically convex open subset of* E. *Then* U *is* $\mathcal{P}_\theta(\mathrm{E})$-*convex iff* U *is a* θ-*Runge domain.*

THEOREM 1.2. *Let* E *be a Banach space with the* SAP. *Let* θ *be a P-holomorphy type and* U *a Runge open subset of* E. *Then* $\mathcal{H}_\theta(\mathrm{U})$ *is dense in* $(\mathcal{H}(\mathrm{U}), \mathcal{T}_0)$

DEFINITION 1.5. $\mathcal{H}_{\mathrm{N}b}(\mathrm{E})$ is the set of all $f \in \mathcal{H}_\mathrm{N}(\mathrm{E})$ such that

$$\left(\left\|\frac{1}{m!}\,\hat{d}f(x)\right\|_\mathrm{N}\right)^{1/m} \to 0 \quad \text{as} \quad m \to \infty \qquad \forall x \in \mathrm{E}.$$

THEOREM 1.3. *Let* E *be a Banach space with the approximation property* (A.P.). *Then* $\mathcal{H}_{\mathrm{N}b}(\mathrm{E})$ *is dense in* $(\mathcal{H}(\mathrm{E}), \mathcal{T}_0)$.

COROLLARY 1. *Let* E *be a Banach space with the* AP. *Then* $(\mathscr{H}_{Nb}(E), \mathscr{T}_0)$ *is complete iff* $\dim E < \infty$.

COROLLARY 2. *Let* E *be a Banach space with the* AP. *Then* $(\mathscr{H}_N(E), \mathscr{T}_0)$ *is complete iff* $\mathscr{H}_N(E) = \mathscr{H}(E)$.

Analogously, $(\mathscr{H}_c(E), \mathscr{T}_0)$ *is complete iff* $\mathscr{H}_c(E) = \mathscr{H}(E)$.

2. SOME RESULTS WHEN θ IS THE CURRENT TYPE

DEFINITION 2.1. A subset U of E′ is $\mathscr{T}$-Runge iff P(E) is dense in $(\mathscr{H}(U), \mathscr{T})$ ($\mathscr{T}$ = topology in $\mathscr{H}(U)$).

DEFINITION 2.2. A subset U of E is $\mathscr{T}$-L-Runge iff $\forall f \in \mathscr{H}(U)$ there exists a net $(P_\alpha)_{\alpha \in \Gamma} \subset \mathscr{P}(E)$ such that $P_\alpha \to f$ in $\mathscr{T}$ and $\{P_\alpha\}_{\alpha \in \Gamma}$ is $\mathscr{T}$-bounded ($\mathscr{T}$ = topology in $\mathscr{H}(U)$).

DEFINITION 2.3. A complex l.c.s. E is holomorphically barrelled if, for any complex l.c.s. F and for any non void subset U of E we have: if $\mathscr{X} \subset \mathscr{H}(U; F)$ is bounded on every finite dimensional compact subset of U, then $\mathscr{X}$ is equi-continuous.

PROPOSITION 2.1. *Let* E *be a holomorphically barrelled locally convex topological vector space and let* U *be an open subset of* E. *Then* U *is* $\mathscr{T}_0$-L-*Runge* $\Leftrightarrow$ U *is* $\mathscr{T}_{0f}$-L-*Runge*.

PROPOSITION 2.2. *Let* E *be a complex Silva space and* U *be an open subset of* E. *Then* U *is* $\mathscr{T}_0$-*Runge* $\Longleftrightarrow$ U *is* $\mathscr{T}_{0f}$-*Runge*.

PROPOSITION 2.3. *Let* E *be a Banach space and* F *a locally convex topological vector space. Let* $U \subset E$ *and* $U' \subset F$ *be holomorphically convex open sets and* $u: E \to F$ *be a holomorphic application of* U *in* F, *continuous on* E. *Then* $U_u = \{z \in U : u(z) \in U'\}$ *is holomorphically convex.*

PROPOSITION 2.4. *Let* E *be the strict inductive limit (see* [11]) *of a sequence* (E_m) *of Banach spaces. We suppose that* E *has an enumerable basis. Let* U *be a holomorphically convex open subset of* E. *Then* U *is an open set of Runge iff* $U_m = \rho_m^{-1}(U)$ *is an open set of Runge in* E_m $\forall m \in \mathbf{N}$.

3. NORMAL CONVERGENCE OF THE TAYLOR SERIES OF $f \in \mathscr{H}(U)$ IN U WHEN U IS A CIRCULAR OPEN CONVEX SET CONTAINING THE ORIGIN

THEOREM 3.1. *Let* U *be a connected circled domain such that* $0 \in U$. *Let* $f \in \mathscr{H}(U)$. *Then* f *can be written uniquely as a series of homogeneous polynomials* $f(z) = \sum_{k=0}^{\infty} P_k(z)$ *on* U (P_k = *homogeneous polynomial of degree* k *in* z) *and the series converges normally.*

COROLLARY 1. *If $f(z)$ is a holomorphic function in a connected circled domain* U *that contains zero, then f can be holomorphically continued to* $\bigcup_{0 \leq |t| \leq 1} (t\mathrm{U})$.

COROLLARY 2. *A connected circled domain* U *that contains zero is a* $\mathscr{C}_0$-L-*Runge domain.*

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