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Francesca Rolandi, Maurizio Verri

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Analisi matematica. — *On the variational inequalities associated to a hydrodynamical model* (*). Nota (**) di FRANCESCA ROLANDI (***) e MAURIZIO VERRI (****), presentata dal Socio L. AMERIO.

RIASSUNTO. — Si dà un teorema di esistenza per le soluzioni di un problema misto relativo a un sistema di disequazioni variazionali di tipo Navier-Stokes e si dimostra poi che, sotto opportune ipotesi, risulta ben posto il corrispondente problema per il sistema di equazioni differenziali associato.

1. In this paper we state a theorem concerning the existence of solutions for a mixed problem associated to a system of Navier-Stokes type variational inequalities occurring in the study of water circulation in a basin. Furthermore, we prove that, under some additional assumptions on the behaviour of the solution in a neighbourhood of $t = 0$, the corresponding problem for the related system of differential equations is well posed, i.e. an existence, uniqueness and continuous dependence theorem holds. Let x, y denote horizontal cartesian coordinates and t the time variable. Then the *basic equations* of the hydrodynamical model we are concerned with, expressing the conservation of mass and momentum of the fluid filling the basin, have the form

$$\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{\xi}\right) + \frac{\partial}{\partial y}\left(\frac{uv}{\xi}\right) = fv + \mu\Delta u - \frac{\xi}{\rho}\left[\frac{\partial p_0}{\partial x} + \rho g\left(\frac{\partial \xi}{\partial x} - \frac{\partial h}{\partial x}\right)\right] + \frac{1}{\rho}\left[U_s - \alpha u\sqrt{u^2 + v^2}\right], \tag{1.2}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{uv}{\xi}\right) + \frac{\partial}{\partial y}\left(\frac{v^2}{\xi}\right) = -fu + \mu\Delta v - \frac{\xi}{\rho}\left[\frac{\partial p_0}{\partial y} + \rho g\left(\frac{\partial \xi}{\partial y} - \frac{\partial h}{\partial y}\right)\right] + \frac{1}{\rho}\left[V_s - \alpha v\sqrt{u^2 + v^2}\right] \tag{1.3}$$

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(***) Istituto di Matematica, Facoltà di Ingegneria del Politecnico di Milano.

(****) Istituto di Matematica, Informatica e Sistemistica dell'Università degli Studi di Udine.

where the wather depth $\xi = \xi(x, y, t)$ from the bottom to the free surface and the linear flows $u = u(x, y, t)$ and $v = v(x, y, t)$ in the x– and y–directions are the unknowns. The atmosferic pressure $p_0 = p_0(x, y, t)$, the height of the bottom $h = h(x, y)$ w.r.t. a horizontal reference plane and the x– and y–components of the surface wind stress $U_s = U_s(x, y, t)$ and $V_s = V_s(x, y, t)$ are given functions. The Coriolis parameter f, the water density ρ, the eddy viscosity μ, the acceleration of gravity g and the bottom frictional coefficient α are given constants.

A number of boundary conditions may be applied to system (1.1)–(1.3), each corresponding to a different physical situation. In this note we shall consider the case when the boundary Γ of the basin is no-slip, i.e. the fluid velocity vanishes at any time on the whole of Γ.

The above model is derived by averaging the exact point (continuity and Navier–Stokes) hydrodynamical equations over the vertical coordinate z [1, 2]. The analysis of the physical assumptions and approximations yielding (1.1)–(1.3) shows [2] that this model cannot be considered as consistent with the basic laws unless the following additional requirements are satisfied at any time throughout the domain Ω under consideration:

$$|\xi(x, y, t)| > \sigma ; \tag{1.4}$$

$$\left|\frac{\partial \xi(x, y, t)}{\partial t}\right| < M_1 , \qquad |\operatorname{grad} \xi(x, y, t)| < M_2 ; \tag{1.5}$$

$$\left|\frac{\partial u(x, y, t)}{\partial t}\right| < M_3 , \qquad |\operatorname{grad} u(x, y, t)| < M_4 ; \tag{1.6}$$

$$\left|\frac{\partial v(x, y, t)}{\partial t}\right| < M_5 , \qquad |\operatorname{grad} v(x, y, t)| < M_6 , \tag{1.7}$$

where σ and the M_i's are suitable positive constants. The general theory of variational inequalities provides a convenient framework for the mathematical treatment of system (1.1)–(1.3) subject to the *consistency conditions* (1.4)–(1.7) (compare [3]), and this is the approach that we will pursue.

2. Let Ω be a bounded open subset of $\mathbf{R}^2$ with boundary Γ, $\boldsymbol{x} = (x, y) \in \Omega$, $Q = \Omega \times]0, T[$ with $0 < T < \infty$. In the sequel, we shall employ standard notations and results concerning Sobolev spaces, for which we refer, e.g., to [4]. If $(\boldsymbol{x}, t) \mapsto \varphi(\boldsymbol{x}, t)$ is a function defined on Q, we write, as usual, $\varphi(t) = \varphi(\cdot, t) = \{\varphi(\boldsymbol{x}, t) \mid \boldsymbol{x} \in \Omega\}$ when it is viewed as a map of $]0, T[$ into some space of functions on Ω, and set $\varphi'(t) = \partial\varphi/\partial t(\cdot, t)$. Furthermore, (spaces of) 2-dimensional vector functions are denoted in boldface and all product spaces are supposed to be equipped with the usual product topology.

Now, define the sets

$$(2.1)\qquad \mathcal{K}_\xi = \left\{ \xi \in L^2(Q) \mid \xi(0) = \xi_0 \ ; \ \xi(\boldsymbol{x}, t) \geq \sigma \ , \ \left| \frac{\partial \xi(\boldsymbol{x}, t)}{\partial t} \right| \leq M_1 \ , \ |\operatorname{grad} \xi(\boldsymbol{x}, t)| \leq M_2 \quad \text{a.e. in } Q \right\},$$

$$(2.2)\qquad \mathcal{K}_u = \left\{ \boldsymbol{u} = (u, v) \in \mathbf{L}^2(Q) \mid \boldsymbol{u}(0) = \boldsymbol{u}_0 \ ; \ \boldsymbol{u}(t)|_\Gamma = \mathbf{0} \quad \text{a.e. in }]0, T[\ ; \ \left| \frac{\partial u(\boldsymbol{x}, t)}{\partial t} \right| \leq M_3 \ , \ |\operatorname{grad} u(\boldsymbol{x}, t)| \leq M_4 \ , \ \left| \frac{\partial v(\boldsymbol{x}, t)}{\partial t} \right| \leq M_5 \ , \ |\operatorname{grad} v(\boldsymbol{x}, t)| \leq M_6 \quad \text{a.e. in } Q \right\}$$

where σ and the M_i's are fixed positive constants, and

$$(2.3)\qquad \xi_0 \in L^\infty(\Omega) \ ; \ \xi_0(\boldsymbol{x}) \geq \sigma \ , \ |\operatorname{grad} \xi_0(\boldsymbol{x})| \leq M_2 \quad \text{a.e. in } \Omega \ ;$$

$$(2.4)\qquad \boldsymbol{u}_0 = (u_0, v_0) \in \mathbf{L}^\infty(\Omega) \ , \ \boldsymbol{u}_0|_\Gamma = \mathbf{0} \ ; \ |\operatorname{grad} u_0| \leq M_4 \quad \text{and} \quad |\operatorname{grad} v_0| \leq M_6 \quad \text{a.e. in } \Omega .$$

Clearly, $\mathcal{K}_\xi$ and $\mathcal{K}_u$ are non-empty, bounded, closed, convex subsets of the Banach space of uniformly Lipschitz continuous (vector) functions in Q. In addition, it can be easily checked that they are closed w.r.t. the $L^2(Q)$–norm.

Next, we set

$$(2.5)\qquad \mathbf{F} = \mathbf{F}(\boldsymbol{x}, t, \xi, \boldsymbol{u}) = (F_1, F_2)$$

where

$$(2.6)\qquad F_1 = \xi \frac{\partial}{\partial x}(p_0 - h) - U_s + u\,|\boldsymbol{u}| - v ,$$

$$(2.7)\qquad F_2 = \xi \frac{\partial}{\partial y}(p_0 - h) - V_s + v\,|\boldsymbol{u}| + u ,$$

and consider the following *variational problem*:

find a pair of functions $(\xi, \boldsymbol{u})$ *such that*

a) $(\xi, \boldsymbol{u}) \in \mathcal{K}_\xi \times \mathcal{K}_u$;

b) $(\xi, \boldsymbol{u})$ *satisfies the following system of variational inequalities*:

$$(2.8)\qquad (\xi' + \operatorname{div} \boldsymbol{u}, \varphi - \xi)_{L^2(Q)} \geq 0 \qquad \forall \varphi \in \mathcal{K}_\xi ;$$

$$(2.9)\qquad \left(\boldsymbol{u}' + (\boldsymbol{u} \cdot \operatorname{grad}) \frac{\boldsymbol{u}}{\xi} + \frac{\boldsymbol{u}}{\xi} \operatorname{div} \boldsymbol{u} + \xi \operatorname{grad} \xi + \mathbf{F}(\xi, \boldsymbol{u}), \psi - \boldsymbol{u}\right)_{\mathbf{L}^2(Q)} + \mu (\boldsymbol{u}, \psi - \boldsymbol{u})_{L^2(0,T;\mathbf{H}_0^1(\Omega))} \geq 0 \qquad \forall \psi \in \mathcal{K}_u .$$

As well known from the theory of variational inequalities, the above problem is formally related to system (1.1)–(1.3) (where, for the sake of brevity, the constants f, ρ, g and α have been set equal to one) subject to *initial conditions*

$$\text{(2.10)} \qquad \xi(x, y, 0) = \xi_0(x, y) \quad , \quad u(x, y, 0) = u_0(x, y), \\ v(x, y, 0) = v_0(x, y) \quad \text{on} \quad \Omega,$$

to *boundary conditions*

$$\text{(2.11)} \qquad u(x, y, t) = v(x, y, t) = 0 \quad \text{on} \quad \Gamma \times]0, T[$$

and to the *consistency conditions* (1.4)–(1.7). In the forthcoming Section we shall express this relationship in a precise form.

Let us now prove the existence of solutions to the variational problem *a*), *b*).

THEOREM 2.1. *Let* Ω *be a bounded open subset of* $\mathbf{R}^2$ *having the cone property, and let* $\mathcal{K}_\xi$, $\mathcal{K}_u$, $\mathbf{F}$ *be defined by* (2.1)–(2.7). *Assume*

$$\operatorname{grad}(p_0 - h) \in \mathbf{L}^2(Q) \quad \text{and} \quad U_S, V_S \in L^2(Q).$$

Then there exists a solution $(\xi, \boldsymbol{u}) \in \mathcal{K}_\xi \times \mathcal{K}_u$ *to the system* (2.8)–(2.9).

Proof. *i*) For fixed $\varepsilon > 0$ and $(\eta, \boldsymbol{v}) \in \mathcal{K}_\xi \times \mathcal{K}_u \equiv \mathcal{K}$, we consider the following two (uncoupled) linear elliptic variational inequalities, whose unknowns are denoted by ξ_ε and $\boldsymbol{u}$, respectively:

$$\text{(2.12)} \qquad (\eta' + \operatorname{div} \boldsymbol{v}, \varphi - \xi_\varepsilon)_{L^2(Q)} + \varepsilon(\xi_\varepsilon, \varphi - \xi_\varepsilon)_{H^1(Q)} \geq 0 \qquad \forall \varphi \in \mathcal{K}_\xi,$$

$$\text{(2.13)} \qquad \left(\boldsymbol{v}' + (\boldsymbol{v} \cdot \operatorname{grad})\frac{\boldsymbol{v}}{\eta} + \frac{\boldsymbol{v}}{\eta} \operatorname{div} \boldsymbol{v} + \eta \operatorname{grad} \eta + \mathbf{F}(\eta, \boldsymbol{v}), \boldsymbol{\psi} - \boldsymbol{u}\right)_{\mathbf{L}^2(Q)} + \\ + \mu(\boldsymbol{u}, \boldsymbol{\psi} - \boldsymbol{u})_{L^2(0,T;\mathbf{H}_0^1(\Omega))} \geq 0 \qquad \forall \boldsymbol{\psi} \in \mathcal{K}_u.$$

Since $\mathcal{K}_\xi$ and $\mathcal{K}_u$ are closed convex subsets of the real Hilbert spaces $H^1(Q)$ and $L^2(0, T; \mathbf{H}_0^1(\Omega))$, respectively, and the inner products $(\cdot, \cdot)_{H^1(Q)}$ and $(\cdot, \cdot)_{L^2(0,T;\mathbf{H}_0^1(\Omega))}$ are coercive bilinear forms, the variational inequalities (2.12) and (2.13) possess unique solutions $\xi_\varepsilon \in \mathcal{K}_\xi$, $\boldsymbol{u} \in \mathcal{K}_u$ [5]. Then, the map $\mathcal{F}_\varepsilon : \mathcal{K} \mapsto \mathcal{K}$, $\mathcal{F}_\varepsilon(\eta, \boldsymbol{v}) = (\xi_\varepsilon, \boldsymbol{u})$ is well defined and one-to-one.

ii) Next, we claim that $\mathcal{F}_\varepsilon$ is continuous w.r.t. the weak topology of $H^1(Q) \times L^2(0, T; \mathbf{H}_0^1(\Omega))$. To this end, let $(\eta_n, \boldsymbol{v}_n) \in \mathcal{K}$ with $w - \lim_{n \to \infty} (\eta_n, \boldsymbol{v}_n) = (\eta, \boldsymbol{v})$ and define the sequence $(\xi_{\varepsilon n}, \boldsymbol{u}_n) \equiv \mathcal{F}_\varepsilon(\eta_n, \boldsymbol{v}_n)$. We have to show that such a sequence converges to $\mathcal{F}_\varepsilon(\eta, \boldsymbol{v})$ weakly. Now, since $\mathcal{K}$ is a bounded weakly closed subset of the Hilbert space $H^1(Q) \times L^2(0, T; \mathbf{H}_0^1(\Omega))$, it is weakly compact; hence, the sequence $(\xi_{\varepsilon n}, \boldsymbol{u}_n)$ will converge in the weak topology if and only if it has exactly one weak limit point. Let $(\bar{\xi}_\varepsilon, \bar{\boldsymbol{u}})$ be one of such limit points, i.e. a subsequence $(\xi_{\varepsilon n_k}, \boldsymbol{u}_{n_k})$ of $(\xi_{\varepsilon n}, \boldsymbol{u}_n)$ is given such that $w - \lim_{n_k \to \infty} (\xi_{\varepsilon n_k}, \boldsymbol{u}_{n_k}) = (\bar{\xi}_\varepsilon, \bar{\boldsymbol{u}})$, and consider the variational inequalities defined by

$$(2.14) \qquad (\xi_{\varepsilon n_k}, \boldsymbol{u}_{n_k}) \equiv \mathscr{F}_\varepsilon (\eta_{n_k}, \boldsymbol{v}_{n_k}) .$$

First of all, we have

$$s - \lim_{n_k \to \infty} \eta_{n_k} = \eta \quad \text{in } L^2(Q) \quad , \quad s - \lim_{n_k \to \infty} \boldsymbol{v}_{n_k} = \boldsymbol{v} \quad \text{in } \mathbf{L}^2(Q)$$

since $\{(\eta_{n_k}, \boldsymbol{v}_{n_k})\} \subset \mathscr{K}$; hence, by using standard convergence arguments [6], we get in the limit as $n_k \to \infty$

$$(2.15a) \qquad (\eta'_{n_k} + \operatorname{div} \boldsymbol{v}_{n_k}, \varphi - \xi_{\varepsilon n_k})_{L^2(Q)} + \varepsilon (\xi_{\varepsilon n_k}, \varphi)_{H^1(Q)} \to$$
$$\to (\eta' + \operatorname{div} \boldsymbol{v}, \varphi - \bar{\xi}_\varepsilon)_{L^2(Q)} + \varepsilon (\bar{\xi}_\varepsilon, \varphi)_{H^1(Q)} \qquad \forall \varphi \in \mathscr{K}_\xi ,$$

$$(2.15b) \qquad \Big(\boldsymbol{v}'_{n_k} + (\boldsymbol{v}_{n_k} \cdot \operatorname{grad}) \frac{\boldsymbol{v}_{n_k}}{\eta_{n_k}} + \frac{\boldsymbol{v}_{n_k}}{\eta_{n_k}} \operatorname{div} \boldsymbol{v}_{n_k} + \eta_{n_k} \operatorname{grad} \eta_{n_k} +$$
$$+ \mathbf{F}(\eta_{n_k}, \boldsymbol{v}_{n_k}), \boldsymbol{\psi} - \boldsymbol{u}_{n_k}\Big)_{\mathbf{L}^2(Q)} + \mu (\boldsymbol{u}_{n_k}, \boldsymbol{\psi})_{L^2(0,T;\mathbf{H}^1_0(\Omega))} \to$$
$$\to \Big(\boldsymbol{v}' + (\boldsymbol{v} \cdot \operatorname{grad}) \frac{\boldsymbol{v}}{\eta} + \frac{\boldsymbol{v}}{\eta} \operatorname{div} \boldsymbol{v} + \eta \operatorname{grad} \eta +$$
$$+ \mathbf{F}(\eta, \boldsymbol{v}), \boldsymbol{\psi} - \bar{\boldsymbol{u}}\Big)_{\mathbf{L}^2(Q)} + \mu (\bar{\boldsymbol{u}}, \boldsymbol{\psi})_{L^2(0,T;\mathbf{H}^1_0(\Omega))} \qquad \forall \boldsymbol{\psi} \in \mathscr{K}_u .$$

Furthermore, taking the weak lower semi-continuity of the Hilbert norm into account, it follows

$$(2.16a) \qquad \liminf_{n_k \to \infty} \varepsilon \| \xi_{\varepsilon n_k} \|^2_{H^1(Q)} \geq \varepsilon \| \bar{\xi}_\varepsilon \|^2_{H^1(Q)} ,$$

$$(2.16b) \qquad \liminf_{n_k \to \infty} \mu \| \boldsymbol{u}_{n_k} \|^2_{L^2(0,T;\mathbf{H}^1_0(\Omega))} \geq \mu \| \bar{\boldsymbol{u}} \|^2_{L^2(0,T;\mathbf{H}^1_0(\Omega))} .$$

Therefore, letting $n_k \to \infty$ in (2.14), we get, by (2.15) and (2.16) and the very definition of $\mathscr{F}_\varepsilon$,

$$(\bar{\xi}_\varepsilon, \bar{\boldsymbol{u}}) = \mathscr{F}_\varepsilon (\eta, \boldsymbol{v})$$

thus proving that $\mathscr{F}_\varepsilon (\eta_n, \boldsymbol{v}_n)$ as a whole converges to $\mathscr{F}_\varepsilon (\eta, \boldsymbol{v})$ in the weak topology as $n \to \infty$.

iii) Since $\mathscr{K}$ is convex and weakly compact in $H^1(Q) \times L^2(0, T; \mathbf{H}^1_0(\Omega))$ and $\mathscr{F}_\varepsilon$ is a weakly continuous map of $\mathscr{K}$, the Schauder–Tychonov theorem applies, hence $\mathscr{F}_\varepsilon$ has a fixed point. Explicitly, there exists $(\xi_\varepsilon, \boldsymbol{u}) \in \mathscr{K}$ such that

$$(2.17) \qquad (\xi_\varepsilon' + \operatorname{div} \boldsymbol{u}, \varphi - \xi_\varepsilon)_{L^2(Q)} + \varepsilon (\xi_\varepsilon, \varphi - \xi_\varepsilon)_{H^1(Q)} \geq 0 \qquad \forall \varphi \in \mathscr{K}_\xi ,$$

$$(2.18) \qquad \left(\boldsymbol{u}' + (\boldsymbol{u} \cdot \operatorname{grad}) \frac{\boldsymbol{u}}{\xi_\varepsilon} + \frac{\boldsymbol{u}}{\xi_\varepsilon} \operatorname{div} \boldsymbol{u} + \xi_\varepsilon \operatorname{grad} \xi_\varepsilon + \right.$$

$$\left. + \mathbf{F}(\xi_\varepsilon, \boldsymbol{u}), \psi - \boldsymbol{u}\right)_{\mathbf{L}^2(Q)} + \mu (\boldsymbol{u}, \psi - \boldsymbol{u})_{L^2(0,T;\mathbf{H}^1_0(\Omega))} \geq 0 \quad \forall \psi \in \mathscr{K}_u .$$

Now, since the net $(\xi_\varepsilon, \boldsymbol{u})$ is bounded in $\mathscr{K}$, one can extract a subnet, still denoted by $(\xi_\varepsilon, \boldsymbol{u})$, which converges weakly to an element $(\xi, \boldsymbol{u}) \in \mathscr{K}$. Passing to the limit in (2.17) and (2.18), we obtain by the same compactness arguments as in *ii*) that the limits ξ and $\boldsymbol{u}$ satisfy (2.8) and (2.9) since the elliptic term in (2.17) vanishes as $\varepsilon \to 0$. QED

Remarks 2.1. The above theorem is also valid in higher dimension ($n \geq 2$). Furthermore, it can be extended to cover the case when $\mathbf{F} = \mathbf{F}(\xi, \boldsymbol{u})$ is any continuous map from $\mathscr{K}$ (endowed with the weak topology of $H^1(Q) \times L^2(0, T; \mathbf{H}^1_0(\Omega))$) into $\mathbf{L}^2(Q)$.

2.2. – The problem of the uniqueness of the solution of *a*), *b*) is open.

3. As already pointed out, the solutions of the variational problem *a*), *b*) can be related to the solutions of the "classical" problem (1.1)–(1.7), (2.10)–(2.11). Indeed, let $(\xi, \boldsymbol{u}) \in \mathscr{K}_\xi \times \mathscr{K}_u$ be a solution of (2.8)–(2.9) and define

$$(3.1) \qquad Q_0 = \{(x, y, t) \in Q \mid \text{ conditions (1.4)–(1.7) hold}\} ,$$

$$(3.2) \qquad T^* = \sup \{t \geq 0 \mid \Omega \times]0, t[\subset Q_0\} ,$$

$$(3.3) \qquad Q^* = \Omega \times]0, T^*[.$$

Then, choosing $\varphi = \xi \pm \lambda f$ and $\psi = \boldsymbol{u} \pm \lambda \boldsymbol{g}$ ($f \in \mathscr{D}(Q^*)$, $\boldsymbol{g} \in \boldsymbol{D}(Q^*)$ and $\lambda > 0$ small enough), inequalities (2.8) and (2.9) yield respectively

$$(3.4) \qquad \xi' + \operatorname{div} \boldsymbol{u} = 0 \quad \text{ in } L^2(Q^*) ,$$

$$(3.5) \qquad \boldsymbol{u}' + (\boldsymbol{u} \cdot \operatorname{grad}) \frac{\boldsymbol{u}}{\xi} + \frac{\boldsymbol{u}}{\xi} \operatorname{div} \boldsymbol{u} + \xi \operatorname{grad} \xi + \mathbf{F}(\xi, \boldsymbol{u}) = \mu \Delta \boldsymbol{u}$$

$$\text{in } L^2(0, T^*; \mathbf{H}^{-1}(\Omega)) .$$

But $(\xi, \boldsymbol{u}) \in \mathscr{K}_\xi \times \mathscr{K}_u$, hence the l.h.s. of (3.5) actually belongs to $\mathbf{L}^2(Q^*)$. Thus, we reach the conclusion that a solution to the variational problem *a*), *b*) satisfies (1.1)–(1.7), (2.10)–(2.11) a.e. in Q^*. It should be noted that, according to our approach, the model loses its physical meaning for $t \geq T^*$ since

$\Omega\times]T^*, T[$ contains a set of positive measure where at least one of the consistency conditions (1.4)–(1.7) does not hold. However, the crucial question concerning the explicit estimate of T^* is still unsolved and seems to be a hard problem.

Now, we shall prove that the above solution in Q^* is actually unique (Theorem 3.1) and, in addition, it depends continuously on the data (Theorem 3.2).

THEOREM 3.1. *Let* Ω *and* $\mathbf{F}$ *be as in Theorem 2.1. Assume*

$$\operatorname{grad}(p_0 - h) \in \mathbf{L}^\infty(Q) \quad \text{and} \quad U_s, V_s \in L^2(Q).$$

Then there exists at most one pair $(\xi, \boldsymbol{u}) \in H^1(Q)\times L^2(0, T; \mathbf{H}_0^1(\Omega))$ *satisfying* (1.1)–(1.7), (2.10)–(2.11), (2.3)–(2.4) *a.e. in* Q.

Proof. Suppose $(\xi_i, \boldsymbol{u}_i)$ $(i = 1, 2)$ are two possible solutions. Then $(\xi, \boldsymbol{u}) = (\xi_1 - \xi_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)$ satisfies the equations (a.e. in Q)

$$\xi' + \operatorname{div}\boldsymbol{u} = 0, \tag{3.6}$$

$$\boldsymbol{u}' - \mu\Delta\boldsymbol{u} + (\boldsymbol{u}_1\cdot\operatorname{grad})\frac{\boldsymbol{u}_1}{\xi_1} - (\boldsymbol{u}_2\cdot\operatorname{grad})\frac{\boldsymbol{u}_2}{\xi_2} + \frac{\boldsymbol{u}_1}{\xi_1}\operatorname{div}\boldsymbol{u}_1 - \tag{3.7}$$
$$-\frac{\boldsymbol{u}_2}{\xi_2}\operatorname{div}\boldsymbol{u}_2 + \xi_1\operatorname{grad}\xi_1 - \xi_2\operatorname{grad}\xi_2 + \mathbf{F}(\xi_1, \boldsymbol{u}_1) - \mathbf{F}(\xi_2, \boldsymbol{u}_2) = \mathbf{0}$$

and the initial and boundary conditions

$$\xi(0) = 0 \quad , \quad \boldsymbol{u}(0) = \mathbf{0}; \tag{3.8}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma\times]0, T[. \tag{3.9}$$

Taking the inner product of (3.6) with $\xi(t)$ in $L^2(\Omega)$, then integrating from zero to $s\in]0, T[$ and using (3.8)-(3.9) yields

$$\|\xi(s)\|^2_{L^2(\Omega)} \leq 4\int_0^s \|\xi(t)\|_{L^2(\Omega)}\|\boldsymbol{u}(t)\|_{\mathbf{H}_0^1(\Omega)}\,dt. \tag{3.10}$$

Similarly, it follows from (3.7) that

$$\|\boldsymbol{u}(s)\|^2_{\mathbf{L}^2(\Omega)} + 2\mu\int_0^s\|\boldsymbol{u}(t)\|^2_{\mathbf{H}_0^1(\Omega)}\,dt \leq \tag{3.11}$$

$$\leq 2\int_0^s\left|\left((\boldsymbol{u}_1\cdot\operatorname{grad})\frac{\boldsymbol{u}_1}{\xi_1} + \frac{\boldsymbol{u}_1}{\xi_1}\operatorname{div}\boldsymbol{u}_1 - (\boldsymbol{u}_2\cdot\operatorname{grad})\frac{\boldsymbol{u}_2}{\xi_2} - \frac{\boldsymbol{u}_2}{\xi_2}\operatorname{div}\boldsymbol{u}_2, \boldsymbol{u}\right)_{\mathbf{L}^2(\Omega)}\right|dt +$$

$$+ 2 \int_0^s |(\xi_1 \operatorname{grad} \xi_1 - \xi_2 \operatorname{grad} \xi_2 , \boldsymbol{u})_{\mathbf{L}^2(\Omega)}| \, dt +$$

$$+ 2 \int_0^s |(\mathbf{F}(\xi_1 , \boldsymbol{u}_1) - \mathbf{F}(\xi_2 , \boldsymbol{u}_2) , \boldsymbol{u})_{\mathbf{L}^2(\Omega)}| \, dt .$$

Now, the following identities hold

$$(3.12) \qquad \left((\boldsymbol{u}_i \cdot \operatorname{grad}) \frac{\boldsymbol{u}_i}{\xi_i} + \frac{\boldsymbol{u}_i}{\xi_i} \operatorname{div} \boldsymbol{u}_i , \boldsymbol{u}\right)_{\mathbf{L}^2(\Omega)} = -\left((\boldsymbol{u}_i \cdot \operatorname{grad}) \boldsymbol{u} , \frac{\boldsymbol{u}}{\xi_i}\right)_{\mathbf{L}^2(\Omega)} \qquad (i = 1 , 2) ;$$

$$(3.13) \qquad (\xi_1 \operatorname{grad} \xi_1 - \xi_2 \operatorname{grad} \xi_2 , \boldsymbol{u})_{\mathbf{L}^2(\Omega)} = -\tfrac{1}{2} (\xi (\xi_1 + \xi_2), \operatorname{div} \boldsymbol{u})_{\mathrm{L}^2(\Omega)}$$

hence, inserting (3.12) and (3.13) into (3.11) and using the consistency conditions, one has by straightforward computation

$$(3.14) \qquad \| \boldsymbol{u}(s) \|^2_{\mathbf{L}^2(\Omega)} + 2 \mu \int_0^s \| \boldsymbol{u}(t) \|^2_{\mathbf{H}^1_0(\Omega)} \, dt \le$$

$$\le C_1 \int_0^s [\| \boldsymbol{u}(t) \|^2_{\mathbf{L}^2(\Omega)} + \| \boldsymbol{u}(t) \|_{\mathbf{L}^2(\Omega)} \| \boldsymbol{u}(t) \|_{\mathbf{H}^1_0(\Omega)} +$$

$$+ \| \xi(t) \|_{\mathrm{L}^2(\Omega)} \| \boldsymbol{u}(t) \|_{\mathbf{H}^1_0(\Omega)} + \| \xi(t) \|_{\mathrm{L}^2(\Omega)} \| \boldsymbol{u}(t) \|_{\mathbf{L}^2(\Omega)}] \, dt$$

where C_1 depends on $\| \xi_0 \|_{\mathrm{L}^\infty(\Omega)}$, $\| \boldsymbol{u}_0 \|_{\mathbf{L}^\infty(\Omega)}$, $\| \operatorname{grad} (p_0 - h \|_{\mathbf{L}^\infty(Q)}$, T , σ and the M_i's.

Finally, we add (3.10) and (3.14) and apply Cauchy's inequality, thus obtaining

$$\| \xi(s) \|^2_{\mathrm{L}^2(\Omega)} + \| \boldsymbol{u}(s) \|^2_{\mathbf{L}^2(\Omega)} \le C_2 \int_0^s [\| \xi(t) \|^2_{\mathrm{L}^2(\Omega)} + \| \boldsymbol{u}(t) \|^2_{\mathbf{L}^2(\Omega)}] \, dt .$$

Therefore, by virtue of the classical Gronwall inequality, we conclude that

$$\xi = 0 \quad , \quad \boldsymbol{u} = \mathbf{0} . \qquad \text{QED}$$

By a similar technique, we can prove the following

THEOREM 3.2. *Let* $(\xi_n , \boldsymbol{u}_n)$, $n \in \mathrm{N}$, *(resp.,* $(\xi , \boldsymbol{u})$*) satisfy Theorem 3.1 with initial value* $(\xi_{0n} , \boldsymbol{u}_{0n})$ *(resp.,* $(\xi_0 , \boldsymbol{u}_0)$*) and coefficients* $\operatorname{grad} (p_{0n} - h_n)$, U_{sn} , V_{sn} *(resp.,* $\operatorname{grad} (p_0 - h)$, U_s , V_s*). Assume*

$$\lim_{n \to \infty} (\xi_{0n} , \boldsymbol{u}_{0n}) = (\xi_0 , \boldsymbol{u}_0) \qquad \text{in } \mathrm{L}^\infty(\Omega) \times \mathbf{L}^\infty(\Omega) ,$$

$$\lim_{n\to\infty} \operatorname{grad}(p_{0n} - h_n) = \operatorname{grad}(p_0 - h) \qquad \text{in} \quad \mathbf{L}^\infty(Q),$$

$$\lim_{n\to\infty} U_{sn} = U_s \quad , \quad \lim_{n\to\infty} V_{sn} = V_s \qquad \text{in} \quad L^2(Q).$$

Then

$$\lim_{n\to\infty} \xi_n = \xi \qquad \textit{in} \quad L^\infty(0, T; L^2(\Omega)),$$

$$\lim_{n\to\infty} \boldsymbol{u}_n = \boldsymbol{u} \qquad \textit{in} \quad L^\infty(0, T; \mathbf{L}^2(\Omega)) \quad \textit{and in} \quad L^2(0, T; \mathbf{H}_0^1(\Omega)).$$

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