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# Rendiconti

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# Note on the behaviour of solutions of a second order nonlinear difference equation

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**Analisi matematica.** — *Note on the behaviour of solutions of a second order nonlinear difference equation.* Nota (*) di Błażej Szmanda, presentata dal Socio G. Zappa.

Riassunto. — Si studia l'equazione non omogenea del secondo ordine alle differenze,

(*) $$\Delta (r_n \Delta u_n) + a_n f(u_n) = b_n$$

nel suo comportamento asintotico. Fra l'altro, si danno condizioni sufficienti per il tendere allo zero di tutte le soluzioni di (*) non oscillatorie.

## 1. Introduction

In the present paper we consider the second order nonlinear difference equation of the form

(1) $$\Delta (r_n \Delta u_n) + a_n f(u_n) = b_n , \qquad n = 0 , 1 , 2 , \cdots ,$$

where $\Delta$ is the forward difference operator i.e. $\Delta v_n = v_{n+1} - v_n$, $\{r_n\}$, $(a_n\}$, $\{b_n\}$ are the real sequences and the following conditions are assumed to hold:

1) $f : R \to R = (-\infty , \infty)$ , $sf(s) > 0$ for $s \neq 0$,

2) $r_n > 0$ for $n \geq n_0 \geq 0$ , $\sum^{\infty} 1/r_n = \infty$.

By a solution of (1) we mean a real sequence $\{u_n\}$ satisfying equation (1) for $n = 0 , 1 , 2 , \cdots$. We consider only such solutions which are nontrivial for all large $n$. A solution of (1) is said to be *nonoscillatory* if it is eventually positive or eventually negative. Otherwise the solution is said to be *oscillatory*. It is said to be bounded if $|u_n| \leq K$ for $n = 0 , 1 , 2 , \cdots$, where K is a positive constant.

The problem of determining sufficient conditions for oscillation of solutions of nonlinear second order difference equations has been studied, for example, in [5–6] (see also references cited in them).

The purpose of this paper is to derive several criteria for the asymptotic behaviour of solutions of equation (1). The results we obtain are the discrete analogues of some theorems for nonlinear differential equations of second order due to Graef-Spikes [2], Bhatia [1], Yeh [7], Kusano-Onose [3]. For some results concerning the oscillatory and asymptotic behaviour of solutions of linear difference equations of second order we refer in particular to recent results of Patula [4] and the references in [4].

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## 2. MAIN RESULTS

THEOREM 1. *Suppose the following conditions are valid*:

(*i*) $a_n \geq \alpha > 0$ for $n \geq n_0$,

(*ii*) $|f(s)|$ *is bounded away from zero if* $|s|$ *is bounded away from zero,*

(*iii*) *the sequence* $\left\{ B_n = \sum_{k=n_0}^{n-1} b_k \right\}$ *is bounded.*

*If* $\{u_n\}$ *is a nonoscillatory solution of* (1), *then* $\lim_{n\to\infty} u_n = 0$.

*Proof.* We write equation (1) as the equivalent system

$$(2) \qquad \begin{aligned} \Delta u_n &= (w_n + B_n)/r_n , \\ \Delta w_n &= - a_n f(u_n) . \end{aligned}$$

Let $\{u_n\}$ be a nonoscillatory solution of (1), say $\{u_n\}$ is eventually positive. The argument if $\{u_n\}$ is eventually negative is similar and will be omitted.
First it is shown that

$$(4) \qquad \liminf_{n\to\infty} u_n = 0 .$$

Suppose not. Then, by 1) and (*ii*), there exist $n_1 \geq n_0$ and a positive constant $C_1$ such that $f(u_n) \geq C_1$ for $n \geq n_1$. From (2) it follows that

$$w_{n+1} - w_{n_1} = - \sum_{k=n_1}^{n} a_k f(u_k) \leq - C_1 \sum_{k=n_1}^{n} a_k \to - \infty \qquad \text{as} \quad n \to \infty .$$

We then have

$$\Delta u_n = (w_n + B_n)/r_n \leq - 1/r_n ,$$

for $n \geq n_2$, for some $n_2 \geq n_1$. This implies that

$$u_n \leq u_{n_2} - \sum_{k=n_2}^{n-1} 1/r_k \to - \infty , \qquad \text{as} \quad n \to \infty ,$$

contradicting the fact that $\{u_n\}$ is eventually positive. From the above argument, we see also that we must have

$$(4) \qquad \sum^{\infty} a_n f(u_n) < \infty .$$

If $\limsup_{n\to\infty} u_n = \gamma > 0$, then there exists a subsequence $\{u_{n_k}\}$, such that $u_{n_k} \to \gamma$, as $k \to \infty$. Hence there is $k_0$ $(n_{k_0} \geq n_0)$, such that $u_{n_k} \geq \gamma/2$ for $k \geq k_0$ and, by (*ii*), $f(u_{n_k}) \geq C_2$ for $k \geq k_0$ where $C_2$ is a positive constant.

Finally, we have

$$\sum^{n_p} a_n f(u_n) \geq \sum_{k=k_0}^{p} a_{n_k} f(u_{n_k}) \geq \alpha C_2 (p - k_0 + 1) \to \infty ,$$

as $p \to \infty$, so that $\sum^{\infty} a_n f(u_n) = \infty$ contradicting (4). This completes the proof.

A close look at the proof of Theorem 1 ensures the validity of the following

THEOREM 2. *Assume that conditions (ii)–(iii) hold and*

$$(iv) \quad a_n \geq 0 \qquad \text{for} \quad n \geq n_0 , \qquad \sum^{\infty} a_n = \infty .$$

*Then every solution* $\{u_n\}$ *of* (1) *is oscillatory or such that* $\liminf_{n\to\infty} |u_n| = 0$.

Note that in the homogeneous case we have the discrete analogue of Bhatia theorem ([1, Theorem 3]).

THEOREM 3. *If* $b_n \equiv 0$ *and conditions (ii), (iv) hold, then all solutions of* (1) *are oscillatory.*

*Proof.* Assume the theorem is false. Then there is a nonoscillatory solution $\{u_n\}$ of (1). Assume that $u_n > 0$ for $n \geq n_0$ (the case $u_n < 0$ can be treated similarly). Then, by 1) and (*iv*), from (1) we obtain $\Delta (r_n \Delta u_n) \leq 0$ for $n \geq n_0$. Now it is easy to see (cfr. for example [5]) that $\Delta u_n \geq 0$ for $n \geq n_0$, so that $\{u_n\}$ is a nondecreasing sequence for $n \geq n_0$. It then follows from (*ii*) that there is a positive constant A such that $f(u_n) \geq A$ for $n \geq n_0$. Thus, by (*iv*), from (1) we have

$$r_n \Delta u_n - r_{n_0} \Delta u_{n_0} \leq - A \sum_{k=n_0}^{n-1} a_k \to - \infty ,$$

as $n \to \infty$, which contradicts the fact that $\Delta u_n \geq 0$ for $n \geq n_0$.

*Remark 1.* More general oscillation criteria for (1) ($b_n \equiv 0$) are contained in [5].

THEOREM 4. *Suppose condition (ii) holds and*

$$(v) \quad a_n > 0 \qquad \textit{for} \quad n \geq n_0 , \qquad \sum^{\infty} a_n = \infty ,$$

$$(vi) \quad \lim_{n\to\infty} b_n / a_n = 0 .$$

*Then every nonoscillatory solution* $\{u_n\}$ *of* (1) *satisfies* $\liminf_{n\to\infty} |u_n| = 0$.

*Proof.* Let $\{u_n\}$ be a nonoscillatory solution of (1), say $u_n > 0$ for $n \geq n_1 \geq n_0$. First observe that $\{u_n\}$ is also a nonoscillatory solution of

$$\Delta (r_n \Delta u_n) + [a_n - b_n / f(u_n)] f(u_n) = 0 , \qquad n \geq n_1 .$$

Suppose that $\liminf_{n\to\infty} u_n > 0$. Then, by 1) and (*ii*), there exists a positive constant A such that $f(u_n) \geq A$ for $n \geq n_1$. Thus by (*vi*), there exists $n_2 \geq n_1$ such that $b_n/a_n f(u_n) < 1/2$ for $n \geq n_2$. This implies that

$$a_n - b_n/f(u_n) = a_n [1 - b_n/a_n f(u_n)] \geq 1/2\, a_n, \qquad n \geq n_2 .$$

So from (*v*) we get

$$\sum^{\infty} [a_n - b_n/f(u_n)] = \infty .$$

Hence, by Theorem 3, we would have that $\{u_n\}$ is oscillatory. A similar argument holds in the case of an eventually negative solution.

THEOREM 5. *Assume that condition* (*v*) *holds and*

(*vii*) $f(s)$ *is continuous for* $s = 0$,

(*viii*) $\liminf_{n\to\infty} \dfrac{\sum_{k=i}^{n} b_k}{\sum_{k=i}^{n} a_k} \geq C > 0$, *for every* $i \geq n_0$.

*Then no solution of* (1) *approaches zero.*

*Proof*. Let $\{u_n\}$ be a solution of (1), which approaches zero. Then, by (*vii*) and 1), there exists $n_1 \geq n_0$ such that $f(u_n) < C/4$ for $n \geq n_1$. Hence, from (1) we have

$$r_{n+1}\,\Delta u_{n+1} - r_{n_1}\,\Delta u_{n_1} \geq -\frac{C}{4} \sum_{k=n_1}^{n} a_k + \sum_{k=n_1}^{n} b_k ,$$

which, by (*viii*), yields

$$(5) \qquad \frac{r_{n+1}\,\Delta u_{n+1}}{\sum_{k=n_1}^{n} a_k} - \frac{r_{n_1}\,\Delta u_{n_1}}{\sum_{k=n_1}^{n} a_k} \geq -\frac{C}{4} + \frac{\sum_{k=n_1}^{n} b_k}{\sum_{k=n_1}^{n} a_k} \geq -\frac{C}{4} + \frac{C}{2} = \frac{C}{4} > 0 ,$$

for all sufficiently large $n$. It follows from (*v*) and (5) that $r_n\,\Delta u_n \to \infty$ as $n \to \infty$ which, in view of 2), leads to the contradictive conclusion that $u_n \to \infty$ as $n \to \infty$.

*Remark* 2. If we replace conditions (*v*) and (*viii*) by

(*v'*) $a_n < 0$ eventually, $\sum^{\infty} a_n = -\infty$,

(*viii'*) $\limsup_{n\to\infty} \left(\sum_{k=i}^{n} b_k\right) \Big/ \left(\sum_{k=i}^{n} a_k\right) \leq C < 0$, $i \geq n_0$,

then the assertion of Theorem 5 holds.

THEOREM 6. *Suppose condition (iii) holds and assume that*

(ix) $\sum^{\infty} a_n^+ = \infty$ , $\sum^{\infty} a_n^-$ *exists, where* $a_n^+ = \max(a_n, 0)$ , $a_n^- = \min(a_n, 0)$ ,

(x) *to every pair of constants* $C_1, C_2$ *with* $0 < C_1 < C_2$ *there corresponds a pair of constants* $M_1, M_2$ *with* $0 < M_1 \le |f(s)| \le M_2$ *for every* $s$ *with* $C_1 \le |s| \le C_2$.

*Then every bounded solution* $\{u_n\}$ *of* (1) *is either oscillatory or such that* $\liminf_{n\to\infty} |u_n| = 0$.

*Proof.* Let $\{u_n\}$ be a bounded nonoscillatory solution of (1). Assume that $\{u_n\}$ is eventually positive. The case $u_n < 0$ is handled similarly. If $\liminf_{n\to\infty} u_n > 0$, then according to (x), there are positive constants $C_1, C_2$ and $n_1 \ge n_0$ such that $C_1 \le u_n \le C_2$ and $M_1 \le f(u_n) \le M_2$ for $n \ge n_1$, where $M_1, M_2$ are also positive constant depending on $C_1, C_2$. Therefore from (1) we obtain

$$r_n \Delta u_n - r_{n_1} \Delta u_{n_1} = -\sum_{k=n_1}^{n-1} a_k^+ f(u_k) - \sum_{k=n_1}^{n-1} a_k^- f(u_k) + \sum_{k=n_1}^{n-1} b_k \le$$

$$\le -M_1 \sum_{k=n_1}^{n-1} a_k^+ - M_2 \sum_{k=n_1}^{n-1} a_k^- + \sum_{k=n_1}^{n-1} b_k .$$

Hence, by use of (iii) and (ix) we have $\lim_{n\to\infty} r_n \Delta u_n = -\infty$ from which, by use of 2), we conclude that $\lim_{n\to\infty} u_n = -\infty$. But this contradicts the fact that $\{u_n\}$ is eventually positive. Thus our assertion is true.

We conclude this paper with the following propositions.

PROPOSITION 1. *Assume that*

(xi) $f(s)$ *is locally bounded in* R,

(xii) $\sum^{\infty} |a_n| < \infty$,

(xiii) $\sum^{\infty} b_n = \infty$.

*Then every solution of* (1) *is unbounded.*

*Proof.* Assume to the contrary that there exists a solution $\{u_n\}$ of (1) which is bounded that is $|u_n| \le K$, where K is a positive constant. By (xi), there exist constants $L_1, L_2$ such that $L_1 \le f(u_n) \le L_2$. Then from (1), by (xii) and (xiii), we obtain

$$r_{n+1} \Delta u_{n+1} - r_{n_0} \Delta u_{n_0} \ge \sum_{k=n_0}^{n} b_k - L_2 \sum_{k=n_0}^{n} a_k^+ - L_1 \sum_{k=n_0}^{n} a_k^- \to \infty ,$$

as $n \to \infty$ i.e. $u_n \to \infty$, a contradiction. Thus the proof is complete.

*Remark 3*. It is clear that Proposition 1 holds if we replace condition (*xiii*) by $\sum^{\infty} b_n = -\infty$.

PROPOSITION 2. *Suppose the following conditions hold*:

(*xiv*) $a_n \geq 0$ *for* $n \geq n_0$,

(*xv*) $f(s)$ *is bounded from above when* $s$ *is bounded from above*,

(*xvi*) $\sum^{\infty} (b_n - Ma_n) = \infty$ *for any* $M > 0$.

*Then all solutions of* (1) *are unbounded above*.

*Proof*. Let $\{u_n\}$ be a solution of (1) such that $u_n \leq K_1$. Then, by (*xv*), there is a positive constant M such that $f(u_n) \leq M$. Therefore we obtain

$$r_n \Delta u_n - r_{n_0} \Delta u_{n_0} \geq \sum_{k=n_0}^{n-1} (b_k - Ma_k) \to \infty,$$

as $n \to \infty$, that is $u_n \to \infty$. But this is a contradiction. A similar argument leads to the following

PROPOSITION 3. *Let condition* (*xiv*) *holds and assume that*

(*xv'*) $f(s)$ *is bounded from below when* $s$ *is bounded from below*,

(*xvi'*) $\sum^{\infty} (b_n + Ma_n) = -\infty$ *for any* $M > 0$.

*Then all solutions of* (1) *are unbounded below*.

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