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Algebra. — *Algebra-Valued Composite Abstract Homogeneous Polynomials.* Nota (*) di NEYAMAT ZAHEER, presentata dal Socio G. ZAPPA.

RIASSUNTO. — Si denoti con E (rispettivamente V) uno spazio vettoriale (risp. una algebra con identità) sopra una campo K, di caratteristica O algebricamente chiuso. Oggetto di questa nota è impiegare i coni circolari nello studio di certi polinomi omogenei astratti composti a valori nell'algebra (a.h.p.) da E in V e ottenere una formulazione più generale del nostro proprio risultato (si veda il Teorema (2.1) in « Trans. Amer. Math. Soc. », 228 (1977), pp. 345–358) ottenuto prima per gli a.h.p. composti da E in K. Tali studi per i polinomi ordinari furono compiuti in passato da Szegö, Cohn, Egerváry, De Bruijn e Zervos, e il teorema di Szegö è stato generalizzato agli a.h.p. da Marden e dall'Autore (cfr. la nota citata sopra). Comunque, tutti questi teoremi « di tipo Szegö », diventano corollari naturali delle nostre attuali formulazioni.

1. INTRODUCTION

Let E and V be vector spaces over the same field K of characteristic zero. A mapping $P : E \to V$ is called [6 pp. 55, 59], [5, pp. 760–763], [13, pp. 52–61], [17], a *vector-valued abstract homogeneous polynomial* (more briefly, *vector-valued a.h.p.*) of degree n if for every $x, y \in E$,

$$P(sx + ty) = \sum_{k=0}^{n} A_k(x, y)\, s^k t^{n-k} \qquad \forall s, t \in K,$$

where the coefficients $A_k(x, y) \in V$ and are independent of s and t for any given x, y in E. For distinction of cases, the adjective ' vector-valued ' in the above definition will be replaced by ' algebra-valued ' or it will be completely dropped according as V is, in particular, an algebra or the field K. We denote by $\mathbf{P}_n^*$ the class of all vector-valued a.h.p.'s of degree n from E to V (even if V is an algebra) and, by $\mathbf{P}_n$, the class of all a.h.p's of degree n from E to K. The nth-*polar* of P is the mapping (see [6, pp. 55, 59] or [5, pp. 762–763] for its existence and uniqueness) $P(x_1, x_2, \cdots, x_n)$ from E^n to V which is a symmetric n-linear form such that $P(x, x, \cdots, x) = P(x)$ for every $x \in E$. The kth-*polar* of P is then defined by

$$P(x_1, \cdots, x_k, x) = P(x_1, \cdots, x_k, x, x, \cdots, x).$$

(*) Pervenuta all'Accademia il 1° ottobre 1980.

From now on we shall assume throughout that K is an algebraically closed field of characteristic zero and that V is an algebra over K with identity (cf. [5, pp. 19–20] or [12, p. 251]). We know ([1, pp. 38–40], [6, pp. 56–57], [11, pp. 248–255]) that such a field K has a maximal ordered subfield K_0 such that $K = K_0(i) = \{a + ib : a, b \in K_0\}$, where $-i^2 = 1$. Let K denote the usual extension of K by adjoining to it an element ω having the properties of *infinity* (see [14]) and let $D(K_\omega)$ denote the class of all *generalized circular regions* of K_ω as introduced by Zervos [19, pp. 353, 373] (see also [15, p. 346]). The definition and other details concerning these regions can be found in [14], [15], or [16] and are not explicitly needed for purposes of the present study.

The following discussion on circular cones and hermitian cones can be seen in [14]. Given a nucleus N of E^2 and a cicular mapping $G : N \to D(K_\omega)$, we define the *circular cone*, relative to N and G, by

$$E_0(N, G) = \cup T_G(x, y),$$

where

$$T_G(x, y) = \{sx + ty \neq 0 : \quad s, t \in K \; ; \; s/t \in G(x, y)\} \tag{1.1}$$

and where the union ranges over all elements $(x, y) \in N$. The work in papers [14]–[18] uses circular cones in the theory of a.h.p.'s and successfully replaces the role of hermitian cones in the earlier works due to Marden [9] and Hörmander [6]. The relationships between such cones is exhibited in the following propositions due to the author [14, pp. 117–119].

PROPOSITION 1.1. *Let* E_1 *be a hermitian cone in* E. *Given a nucleus* N *of* E^2, *there exists a circular mapping* $G : N \to D(K_\omega)$ *such that* $E_0(N, G) = E_1$ *and* $E_1 \cap \mathscr{L}[x, y] = T_G(x, y)$ *for every* $(x, y) \in N$, *where* T_G *is as defined by* (1.1) *and* $\mathscr{L}[x, y]$ *is the subspace generated by* x *and* y.

PROPOSITION 1.2. *The class of all circular cones in* E *contains properly the class of all hermitian cones in* E.

A mapping $L : V \to K$ (V being an algebra with identity over K) is called [12, p. 253] (see also [17] or [18]) a *scalar homomorphism* on V if it satisfies the following two conditions:

(*i*) $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v) \qquad \forall u, v \in V, \quad \alpha, \beta \in K$;

(*ii*) $L(uv) = L(u) L(v) \qquad \forall u, v \in V$.

Ideal maximal subspaces [12, p. 252] of V are characterized (cf. [12, Theorem 2, p. 254]). in a one-to-one manner, by sets of the form

$$\{v \in V : L(v) = 0\} = L^{\perp} \text{ (say)}, \tag{1.2}$$

where L is a nontrivial scalar homomorphism on V (see also [17] or [18]).

2. SZEGÖ–TYPE THEOREMS

In order to discuss the concrete mathematical problem and to formulate a concise theorem, we need the following concepts. Let $P, Q \in \mathbf{P}_n^*$ and be given by

$$P(s\xi + t\eta) = \sum_{k=0}^{n} C(n, k) A_k(\xi, \eta) s^k t^{n-k} \tag{2.1}$$

$$Q(s\xi + t\eta) = \sum_{k=0}^{n} C(n, k) B_k(\xi, \eta) s^k t^{n-k}, \tag{2.2}$$

where $C(n, k) = n!/(n-k)!$. We define a unique algebra-valued a.h.p. $R \equiv P \wedge Q \in \mathbf{P}_{2n}^*$ by

$$R(s\xi + t\eta) = (P \wedge Q)(s\xi + t\eta) = \sum_{k=0}^{n} C(n, k) A_k(\xi, \eta) B_k(\xi, \eta) s^{2k} t^{2(n-k)}, \tag{2.3}$$

for every $s, t \in K$ and call $P \wedge Q$ as an *algebra-valued $\wedge$-composite* a.h.p. obtained from P and Q. This concept is an immediate extension of the corresponding known idea (see [15, p. 348]) in respect to the class $\mathbf{P}_n$, in which case it will be termed simply as a *$\wedge$-composite* a.h.p. (from E to K). If $L (\not\equiv 0)$ is a scalar homomorphism on V, we define the mapping $LP : E \to K$ by $(LP)(x) = L(P(x))$ for $x \in E$. If $P, Q \in \mathbf{P}_n^*$, it is trivial to see that $LP, LQ \in \mathbf{P}_n$ and that $L(P \wedge Q) = LP \wedge LQ$ (cf. (2.1)–(2.3)).

The concept of supportable subsets of a vector space is well-known [6, p. 59]. The analogous concept in an algebra is given in the following.

DEFINITION 2.1. ([17], [18]). A subset M of V is called *fully supportable* if every point ξ outside M is contained in some ideal maximal subspace of V which does not intersect M. In other words (cf. (1.2)), for every $\xi \in V - M$, there exists a nontrivial scalar homomorphism L on V such that $L(\xi) = 0$ but $L(v) \neq 0$ for every $v \in M$.

Obviously, a fully supportable subset of V is also a supportable subset of V when V is regarded as a vector space, but not conversely. The following proposition gives a general method for constructing a fully supportable subset of an arbitrary algebra V.

PROPOSITION 2.2 ([17], [18]). *The complement in* V *of every ideal maximal subspace of* V *is a fully supportable subset of* V.

If $P \in \mathbf{P}_n^*$ and M is a fully supportable subset of V, we shall write (for given $x, y \in E$)

$$E_P(x, y) = \{sx + ty \neq 0 : \quad s, t \in K \quad ; \quad P(sx + ty) \notin M\}. \tag{2.4}$$

If $V = K$, the only nontrivial scalar homomorphism on V is the identity map from K to K. In this case, Proposition 2.2 implies that $K - \{0\}$ is the only fully supportable subset of K and that the corresponding set $E_P(x, y)$ reduces essentially to the nullset [14] $Z_P(x, y)$ of P given by

$$Z_P(x, y) = \{sx + ty \neq 0 : \; s, t \in K \; ; \; P(sx + ty) = 0\} .$$

In this section, we shall employ the concepts of circular cones and fully supportable sets in the study of algebra-valued $\wedge$-composite a.h.p.'s from E to V and obtain a more general formulation (to such polynomials) of the following result due to the author [15, Theorem (2.1)], established earlier for $\wedge$-composite a.h.p.'s from E to K.

THEOREM 2.3. *Let* $E_0(N, G)$ *be a circular cone in* E *and let* $P, Q \in \mathbf{P}_n$ *such that* $Z_P(x, y) \subseteq T_G(x, y)$ *for some* $(x, y) \in N$. *If* $\mu x + \nu y \in Z_{P \wedge Q}(x, y)$, *then there exist elements* $\alpha x + \beta y \in T_G(x, y)$ *and* $\gamma x + \delta y \in Z_Q(x, y)$ *such that* $\sigma^2 = -\rho\Delta$, *where* $\sigma = \mu/\nu$, $\rho = \gamma/\delta$ *and* $\Delta = \alpha/\beta$ *(when* σ^2 *is of the form* $0 \cdot \omega$ *(resp.* $\omega \cdot 0$*), the equation* $\sigma^2 = -\rho\Delta$ *is to be interpreted as* $-\sigma^2/\rho = \Delta = \omega$ *(resp.* $-\sigma^2/\Delta = \rho = 0$*) or as* $-\sigma^2/\Delta = \rho = 0$ *(resp.* $-\sigma^2/\rho = \Delta = 0$*) according as* $\sigma \neq 0$ *or* $\sigma = 0$*).*

It was shown [15, § 2] that the above theorem generalizes Szegö's theorem (see [10, § 2, Theorem 2] or [7, Theorem (16.1)]) to a.h.p.'s, that it includes the earlier generalizations due to Zervos [19, p. 363] and to Marden [9, Theorem (3.2)], and that it also includes some related results (in the complex plane) due to Cohn [3], de Bruijn [2], and Egerváry [4]. We now give our main theorem of this paper.

THEOREM 2.4. *Let* $E_0(N, G)$ *be a circular cone in* E, M *a fully supportable subset of* V, *and let* $P, Q \in \mathbf{P}_n^*$ *such that* $E_P(x, y) \subseteq T_G(x, y)$ *for some* $(x, y) \in N$. *If* $\mu x + \nu y \in E_{P \wedge Q}(x, y)$, *then there exist elements* $\alpha x + \beta y \in T_G(x, y)$ *and* $\gamma x + \delta y \in E_Q(x, y)$ *such that* $\sigma^2 = -\rho\Delta$, *where* $\sigma = \mu/\nu$, $\rho = \gamma/\delta$, *and* $\Delta = \alpha/\beta$.

Remark. Same interpretations are to be made as in Theorem 2.3 for cases when σ^2 is of the form $0 \cdot \omega$ or $\omega \cdot 0$.

Proof. Suppose that $\mu x + \nu y \in E_R(x, y)$, where $R = P \wedge Q$. Then (cf. (2.4)) $R(\mu x + \nu y) \notin M$. By definition of fully supportable subsets, there exists a nontrivial scalar homomorphism on V such that $L(R(\mu x + \nu y)) = (LR)(\mu x + \nu y) = 0$ but $L(v) \neq 0$ for every $v \in M$. Consequently, $\mu x + \nu y \in Z_{LR}(x, y)$. But we have seen that $LP, LQ \in \mathbf{P}_n$ and $LR = LP \wedge LQ$. Next, we claim that $Z_{LP}(x, y) \subseteq E_P(x, y)$. For if $L(P(sx + ty)) = 0$, the property of L implies that $P(sx + ty) \notin M$ and (hence) $sx + ty \in E_P(x, y)$. From the hypothesis on P we have that $Z_{LP}(x, y) \subseteq T_G(x, y)$. Since all the hypothesis of Theorem 2.3 are now satisfied by LP and LQ and since $\mu x + \nu y \in Z_{LR}(x, y)$, Theorem 2.3 implies the existence of elements $\alpha x + \beta y \in T_G(x, y)$ and $\gamma x + \delta y \in Z_{LQ}(x, y)$ such that $\sigma^2 = -\rho\Delta$ (with the same interpretations as in Theorem 2.3). Finally, since $Z_{LQ}(x, y) \subseteq E_Q(x, y)$, the theorem is proved.

The above theorem reduces essentially to Theorem 2.3 in the case when $V = K$ (see the discussion following the relation (2.4)). Furthermore, Theorem 2.4 furnishes the following result in terms of hermitian cones.

COROLLARY 2.5. *Let* E_1 *be a hermitian cone in* E, M *a fully supportable subset of* V, *and let* $P, Q \in \mathbf{P}_n^*$ *such that* $E_P(x, y) \subseteq E_1$ *for some linearly independent elements* $x, y \in E$. *If* $\mu x + \nu y \in E_{P \wedge Q}(x, y)$, *then there exist elements* $\alpha x + \beta y \in E_1$ *and* $\gamma x + \delta y \in E_Q(x, y)$ *such that* $\sigma^2 = -\rho\Delta$, *where* $\sigma = \mu/\nu$, $\rho = \gamma/\delta$, and $\Delta = \alpha/\beta$ (with same interpretations as in Theorem 2.3).

Proof. Let us take a nucleus N of E^2 such that $(x, y) \in N$ (it is possible). By Proposition 1.1, there exists a circular cone $E_0(N, G)$ such that $E_0(N, G) = E_1$ and

$$E_1 \cap \mathscr{L}[x', y'] = T_G(x', y') \qquad \forall (x', y') \in N. \tag{2.5}$$

From the hypothesis on $E_P(x, y)$ and the relation (2.5) we conclude that $E_P(x, y) \subseteq T_G(x, y)$. By Theorem 2.4 and the relation (2.5), we get elements $\alpha x + \beta y \in E_1$ and $\gamma x + \delta y \in E_Q(x, y)$ such that $\sigma^2 = -\rho\Delta$. This completes our proof.

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