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Natural limitations of algorithmic procedures in logic

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Logica matematica. — *Natural limitations of algorithmic procedures in logic.* Nota (*) di DANIELE MUNDICI, presentata dal Socio G. ZAPPA.

RIASSUNTO. — I più semplici aspetti (quantistici e relativistici) dei procedimenti di calcolo vengono matematizzati: quindi si ricavano risultati limitativi per quanto riguarda (*i*) la realizzabilità pratica dell'interpolazione di Craig, e (*ii*) la decidibilità pratica dell'aritmetica con quantificatori limitati.

0. INTRODUCTION

We give a rigorous formulation of the simplest physical features of (computations by) Turing machines; we imagine the latter are materialized as real computers subject to the laws of Nature: notably irreversibility and uncertainty of time-energy and relativity. We obtain various limitative results concerning the power of algorithmic methods for such logical operations as (*i*) writing down Craig's interpolants for first-order valid implications, and (*ii*) deciding the truth of statements in arithmetic with bounded quantifiers.

The reader might consult [2], [3] or [8] for the necessary background in logic, [1] or [15] for computation theory and [6] for Quantum Mechanics: actually, our definitions below are based on very simple physical concepts which can be found in any textbook.

For the important interpolation and definability properties such as Craig's interpolation and their syntactic and algebraic aspects, see [2], [3], [9–11]; in this paper only the syntactic aspects are considered, much along the lines of [11]. For arithmetic with bounded quantifiers for short Δ-*arithmetic*, see [13]. We let throughout $\mathbf{R}^+ = \{x \in \mathbf{R} \mid x > 0\}$, where $\mathbf{R}$ are the real numbers; for φ a sentence, $\|\varphi\|$ is its length, i.e. the number of (occurrences of) symbols in φ.

1. QUANTUM MECHANICS VERSUS CRAIG'S INTERPOLATION

In [12] we introduced the following definition:

DEFINITION 1.1. *Let* $R = (r_0, \cdots, r_n)$ *be the record of a computation (i.e. a sequence of configurations) with n steps,* $s_1 = (r_0, r_1) \cdot \cdots, s_n = (r_{n-1}, r_n)$; *then* a time-energy measurement M over R *is defined by*

$$M = (t, E, \hbar, \Delta t, \Delta E),$$

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where $t, \mathrm{E}, \hbar \in \mathbf{R}^+$, $\Delta t : (s_1, \cdots, s_n) \to \mathbf{R}^+$, $\Delta \mathrm{E} : (s_1, \cdots, s_n) \to \mathbf{R}^+$ *obey the following conditions*:

(*i*) *irreversibility*: $t \geq \sum_{k=1}^{n} \Delta t(s_k) \quad ; \quad \mathrm{E} \geq \sum_{k=1}^{n} \Delta \mathrm{E}(s_k)$;

(*ii*) *uncertainty*: $\Delta t(s_k) \cdot \Delta \mathrm{E}(s_k) \geq \hbar \qquad (k = 1, \cdots, n)$;

t, E *and* $\hbar$ *are called respectively the* duration, energy *and* quantum constant *of* M, *while* Δt *and* $\Delta \mathrm{E}$ *are called the* time *and* energy function *respectively*.

Remark 1.2. The above definition is motivated by an examination of the basic quantum mechanical features of non-parallel computations: as a matter of fact, if R is thought of as being performed by a real computer, each step s_k will be carried over during a time interval $\Delta t(s_k)$ at the expense of an amount of energy $\Delta \mathrm{E}(s_k)$; if t and E are the total time and energy required by R, then the fact that no portion of the time and energy used for one step can be used again for another step can be simply expressed by the above irreversibility clause (*i*); also, the fact that during $\Delta t(s_k)$ the physical properties of the scanned square are modified, is to the effect that the energy uncertainty $\partial \mathrm{E}_k$ of such physical system must be greater than $\hbar/\Delta t(s_k)$, by Heisenberg's uncertainty law: this is only made possible if the energy $\Delta \mathrm{E}(s_k)$ received by the scanned square is greater than $\partial \mathrm{E}_k$, which is just the content of clause (*ii*) in the above definition.

The above machinery can be applied to the study of the practical feasibility of Craig's interpolation (see [12] for another application); recall that for any valid first-order implication $\varphi \to \psi$, Craig's interpolation theorem asserts the existence of an " interpolant " χ with $\varphi \to \chi$ and $\chi \to \psi$ and such that the only " primitive notions " i.e. the only non-logical symbols occurring in χ are those occurring in both φ and ψ; see [9–11], [8], [3] and [2] for further information of this fundamental logical property.

THEOREM 1.3. *We can explicitly write down a valid first-order implication* $\varphi \to \psi$ *with* $\|\varphi\|, \|\psi\| < 1145$ *whose shortest interpolant* χ, *if actually written down as the outshot of a computation* R, *shall require an amount of time-energy*

$$t \cdot \mathrm{E} > \hbar \cdot 2^{2^{2^{2^{2^{2}}}}}$$

whatever R *and whatever time-energy measurement* $\mathrm{M} = (t, \mathrm{E}, \hbar, \Delta t, \Delta \mathrm{E})$ *over* R.

Proof. First notice that any M over $\mathrm{R} = (r_0, \cdots, r_n)$ satisfies

$$t \cdot \mathrm{E} \geq \hbar \cdot n^2$$

as can be directly proved by using (*i*) and (*ii*) in Definition 1.1 together with the familiar Schwartz inequality (see also [12]]); now observe that, since χ

is the outshot of R, then the number of steps n satisfies $n \geq \| \chi \|$; finally, recall the example of a valid implication $\varphi \rightarrow \psi$ given (for $m = 3$) in the second main theorem in [11], where $\|\varphi\|, \|\psi\| < 1145$ but for any interpolant χ,

$$\| \chi \| > 2^{\cdot^{\cdot^{2}}} \Big\updownarrow \text{ height } 7 .$$

Remark 1.4. The transparent physical content of Theorem 1.3 above is that there exists a very short first-order valid implication (whose length is one page or so) whose Craig's interpolants all require too much time-energy to be explicitly written down; as a matter of fact, in the *mks* system of physical units (where Planck's constant $h = 2\pi\hbar$ is well known to exceed $6 \cdot 10^{-34}$) writing down any such interpolant would require much more than, say,

$$2^{2^{2^{10000}}} \text{ sec} \cdot \text{joule} ;$$

of course, the above amount of time-energy is far beyond the possibilities of mankind, and is likely to remain so for a long time.

2. RELATIVITY VERSUS Δ-ARITHMETIC

In [13] we introduced the following:

DEFINITION 2.1 Let T be an arbitrary Turing machine with z states; then a *space-time expansion* $\mathfrak{M}$ of T is defined by

$$\mathfrak{M} = (\mathrm{T}, \nu, c, \mathfrak{B}),$$

where $\nu, c \in \mathbf{R}^+$ are called the *frequency* and *speed* of $\mathfrak{M}$ respectively, $\mathfrak{B} = \{\mathrm{B}_i\}_{1 \leq i \leq z}$ is a set of pairwise disjoint bounded Borel subsets of $\mathbf{R}^3$, each B_i having a volume $\mathrm{V}_i > 0$; we call $\mathrm{V} = \mathrm{V}_1 + \cdots + \mathrm{V}_z$ the *volume* and $v =$ least among the V_i the *state volume* of $\mathfrak{M}$; if $\mathrm{R} = (r_0, \cdots, r_n)$ is a record of a computation by T with n steps, then $t = n/\nu$ is called the *duration* of R; $\mathfrak{M}$ is called *natural* if it obeys the following condition:

(*relativity*): $\quad$ diameter of $(\mathrm{B}_1 \cup \cdots \cup \mathrm{B}_z) \leq 2\, c/\nu$.

Remark 2.2. The above definition is motivated by an examination of the simplest relativistic features of computations by T, once T materializes as a real computer: essentially, we require that (*i*) the states of T occupy some space, (*ii*) the scanning head of T interacts with the states of T by signals which do not travel at infinite speed, and (*iii*) any two computation steps of T may be safely assumed to have equal duration.

The above machinery is applied in [13] to obtain the following result:

THEOREM 2.3. *Let Turing machine* T *give a decision procedure for* Δ*-arithmetic; let* $\mathfrak{M} = (\mathrm{T}, \nu, c, \mathfrak{B})$ *be an arbitrary space-time expansion of* T *with state volume* v*; let* t *be the sup of the durations of computations by* T *of* Δ*-arithmetical problems* φ *with* $\|\varphi\| \leq j$ *; then*

$$t \geq (j/c) \cdot [(3\,v/4\,\pi) \cdot (j/(373 \cdot \log j) - 3000)]^{1/3}.$$

Remark 2.4. The transparent physical meaning of the above theorem is the existence of a fundamental limitation in the length $j(t)$ of problems which can be mechanically solved in less than t seconds; for example (with respect to the *mks* system) for any Turing machine T deciding Δ-arithmetic there exists a hard problem φ_{T} with less than 30 million symbols which cannot be solved in less than 10^{-11} seconds; similarly, there are hard problems of lenght 10^{23} which cannot be solved in less than 800 years. To obtain such results one considers that in the real world signals cannot travel faster than $c =$ speed of light; one further assumes that the state volume v is greater than $(4/3)\,\pi a^3$ where $a =$ Bohr radius (i.e. each materialized state of T is greater than the hydrogen atom), and directly applies the lower bound given by theorem 2.3 above.

Another consequence of the above theorem is that we are enabled to say something new about the properties of *time* itself, by simply observing computations of Δ-arithmetical problems; for example, if each cycle of a periodic phenomenon has a duration t larger than the duration of the computation by T of the hardest (for T) problems of length $j \leq 10^{15}$ then we can safely conclude that $t \geq 0.6$ seconds, without using a chronometer. One might ask at this point to which extent computers are to time as thermal engines are to temperature.

Remark 2.5. Many asymptotic complexity results can be found in the literature (see, e.g., [4], [5], [7], [14], [16]): here the adjective "asymptotic" is to stress the fact that all these complexity results deal with suitably large sentences, and do not take care of (relatively) short ones: contrast with 1.3. In addition, the results presented in this note show how the "complexity" of logical operations may be translated into the language of physics, thus linking the notion of unfeasibility to specific fundamental limitations arising from physical laws.

REFERENCES

[1] A. V. AHO, J. E. HOPCROFT, J. D. ULLMAN (1974) – *The design and analysis of computer algorithms*, Addison-Wesley, Mass.

[2] J. L. BELL, M. MACHOVER (1977) – *A course in mathematical logic*, North-Holland, Amsterdam.

[3] C. C. CHANG, H. J. KEISLER (1977) – *Model theory*, second ed., North-Holland, Amsterdam.

[4] J. FERRANTE, C. W. RACKOFF (1979) – *The computational complexity of logical theories*, Lecture Notes in Math., *718*, Springer, Berlin.

[5] M. J. FISCHER, M. O. RABIN (1974) – *Super-exponential complexity of Presburger arithmetic*, « SIAM-AMS Proceedings », 7, 27–41.

[6] S. P. GUDDER (1979) – *Stochastic methods in quantum mechanics*, North-Holland, Amsterdam.

[7] A. R. MEYER (1974) – *The inherent complexity of theories of ordered sets*, in: « Proc. Int. Cong. Math. », Vancouver, *2*, Canadian Math. Congress, 477–482.

[8] J. D. MONK (1976) – *Mathematical logic*, Springer, Berlin.

[9] D. MUNDICI (1982) – *Compactness = JEP in any logic*, « Fund. Math. », to appear.

[10] D. MUNDICI (1981) – *Robinson's consistency theorem in soft model theory*, « Trans. AMS », 263, 231–241.

[11] D. MUNDICI – *Complexity of Craig's interpolation*, « J. Symb. Logic », to appear.

[12] D. MUNDICI – *Impracticable theorem-proving Turing machines*, to appear.

[13] D. MUNDICI – *Natural limitations of decision procedures for arithmetic with bounded quantifiers*, to appear.

[14] M. O. RABIN (1977) – *Decidable theories*, in: *Handbook of math. logic* (editor J. Barwise) North-Holland, Amsterdam, 595–630.

[15] J. E. SAVAGE (1976) – *The complexity of computing*, Wiley, New York.

[16] L. J. STOCKMEYER (1974) – *The complexity of decision problems in automata theory and logic*, Proj. MAC Tech. Report, 133.