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Misura e integrazione. — *Abstract absolute Riemannian integration.* Nota (*) di HANS GÜNZLER, presentata dal Socio L. AMERIO.

RIASSUNTO. — Si introducono integrali propri e impropri per funzioni $:X \to S =$ semigruppo commutativo con metrica generalizzata rispetto a una biaddiva $\Phi : S \times \Omega \to S'$ di variazione finita; Ω è un semianello di insiemi $\subset X$. Per il caso proprio si usa una riformulazione della definizione di Riemann, per l'improprio si applica convergenza locale in misura assieme a convergenza in media.

Integration with respect to finitely additive set functions is needed frequently in functional analysis and related fields ([23], [13]). Here we want to consider such an integral for which with Banach-space-valued f also $x \to \| f(x) \|$ is integrable, i.e. an unconditional integral. For broad applicability the range spaces should be as general as possible; since the main problems are closure under addition and additivity of the integral, abelian semigroups would be a natural setting. Such a theory is indeed possible, neither a cancellation law nor convexity are needed. Subsumed are thus integrals for set-valued functions [6], [2], values in a ring with (non-archimedean) valuation, in topological groups, in partially ordered semigroups [4], [7]. By (29) the Lebesgue-Bochner integral is also a special case. Needing some sort of order, we work with semimetrics which are sufficiently flexible (examples 3, 5, 6, 8, 9, 10). Even for Banach-space-values, where we introduced this integral in [11], it contains p.e. strictly the integrals in [1], [3], [8], [19], [20], [27], [31], [32]. Non-absolute integrals are not considered here, for references see [12], p. 2. Also we do not use integral norms as introduced in [1], [5], [29], since we do not want to assume order completeness and lattice structure for the range space of the semimetric; under such assumptions proper integration by integral norms is obvious, by [16], [30] also the improper case can be treated for Banach-space-values. Additional details, examples, references and some generalizations the interested reader can find in [12].

1. STEP FUNCTION AND ELEMENTARY INTEGRALS

If X is a set, $\mathbf{P}(\mathbf{X}) := \{M : M \subset X\}$, then Ω is a **semiring in X** iff $\phi \in \Omega \subset P(X)$ and $U, V \in \Omega$ imply $U \cap V \in \Omega$ and the existence of a natural n and disjoint $A_1, \cdots, A_n \in \Omega$ with $U - V = A_1 \cup \cdots \cup A_n$, with $U - V := \{x \in U : x \notin V\}$. $(S, +, 0)$ is an **abelian semigroup with 0** or **ASZ** if $0 \in S$, $+ : S \times S \to S$ is associative and commutative, with

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$a+0=a$ for $a \in S$; $\boldsymbol{na} := \sum_1^n a$, $0a := 0$ for $a \in S$, n natural. If $f \in S^X := \{f : X \to S\}$, M , P $\subset$ X, then $\boldsymbol{f}\mathbf{M} : X \to S$ is defined by

(1) $(fM)(x) := f(x)$ if $x \in M$, $(fM)(x) := 0$ if $x \in X - M$;

specifically $aM = a$ on M , $= 0$ on $X - M$, for $a \in S$; $MP := M \cap P$. $\mathbf{T}(\boldsymbol{\Omega}, \mathbf{S}) := \{h \in S^X : h = \sum_1^n a_m A_m$ with n natural, $a_m \in S$, $A_m \in \Omega\}$, with pointwise $=$, $+$ (**Ω-step-functions**).

If Ω is a semiring, S an ASZ, then $T(\Omega, S)$ is an ASZ; if $a \in S$, $h \in T(\Omega, S)$, $M \in \boldsymbol{r\Omega} :=$ ring generated by Ω in P(X), then aM, $hM \in T(\Omega, S)$. If S' is also an ASZ and $h \in T(\Omega, S)$, $k \in T(\Omega, S')$, then there are a natural n, disjoint $B_m \in \Omega$, $b_m \in S$, $b'_m \in S'$ with

(2) $h = b_1 B_1 + \cdots + b_n B_n$, $k = b'_1 B_1 + \cdots + B'_n B_n$.

If $h = \sum_1^m a_i A_i$, $k = \sum_1^m a'_i A_i$, there are disjoint $B_j \in \Omega$ with $\bigcup_1^m A_i = \bigcup_1^n B_j$ and $A_i B_j \neq \phi \Rightarrow B_j \subset A_i$; $b_j := \sum_1^m a_{ij}$ with $a_{ij} := a_i$ if $B_j \subset A_i$, else $a_{ij} := 0$, and similarly defined b'_j, do it.

A $\mu : \Omega \to S = $ ASZ is **additive in $\boldsymbol{\Omega}$** iff $\mu(\phi) = 0$ and $\mu(A) = \sum_1^n \mu(A_m)$ whenever A , $A_m \in \Omega$ and $A = A_1 \cup \cdots \cup A_n$ disjoint; $\Phi : S \times \Omega \to S' = $ ASZ is **biadditive** iff Φ_a is additive in Ω , $a \in S$, and Φ_A is additive in S , $A \in \Omega$; here $\boldsymbol{\Phi_a}(\mathbf{A}) := \Phi(a, A) =: \boldsymbol{\Phi_A}(\boldsymbol{a})$, $\varphi : S \to S'$ is **additive in S** iff $\varphi(0) = 0$ and $\varphi(a+b) = \varphi(a) + \varphi(b)$, a , $b \in S$.

LEMMA 1. *If Ω is a semiring in* X, S *and* S' *are* ASZ*s*, $\Phi : S \times \Omega \to S'$ *is biadditive, then*

(3) $$\Phi(h) := \int h \, d\Phi := \sum_1^m \Phi(a_i, A_i) \quad \textit{if} \quad h = \sum_1^m a_i A_i \in T(\Omega, S)$$

is well defined and additive in $T(\Omega, S)$.

So any additive $\mu_0 : \Omega \to S_0 = $ ASZ can be extended uniquely to an additive $\mu : r\Omega \to S$ $\left(\text{use } S = \{0, 1, \cdots\}, S' = S_0, \Phi(n, A) = \sum_1^n \mu_0(A)\right)$.

Proof. If $\sum_1^m a_i A_i = \sum_1^n b_j B_j$, one has to show $\sum_1^m \Phi(a_i, A_i) = \sum_1^n \Phi(b_j, B_j)$. With (2) one can assume the b_j , B_j as after (2). Then

$$\sum_1^m \Phi(a_i, A_i) = \sum_1^m \sum_1^n \Phi(a_i, A_i B_j) = \sum_1^n \sum_1^m \cdots = \sum_1^n \Phi(b_j, B_j),$$

since $\Phi(a_i, A_i B_j) = \Phi(a_{ij}, B_j)$. □

$(I, \leq)$ is **semiordered** iff I is a set, $\leq$ defined in some $M \subset I \times I$, with $a \leq a$, and $a \leq b, b \leq c \Rightarrow a \leq c$, for $a, b, c \in I$; $a < b$ iff $a \leq b$ but not $b \leq a$. $(I, \leq)$ is a **net** iff it is semiordered and to any $a, b \in I$ exists $c \in I$ with $c \leq a, c \leq b$. S or $(S, +, 0, \leq)$ is a **partially ordered abelian semigroup with 0** or **PASZ** iff $(S, +, 0)$ is an ASZ, $(S, \leq)$ semiordered and $a \leq b \Rightarrow a + c \leq b + c$ for $a, b, c \in S$. Then $\mathbf{S_+} := \{a \in S : 0 \leq a\}$, $\mathbf{S^+} := \{a \in S : 0 < a\}$, and if $a, b, c \in S$,

(4) $$d(a, b) \leq c \quad \text{means} \quad a \leq b + c.$$

If V, V' are PASZ*s*, $P \subset V'_+$, then $\varphi : V \to V'$ is **P-monotone** iff $\varphi(u) \leq \varphi(v) + \varepsilon$ for $\varepsilon \in P$, $u, v \in V$ with $u \leq v$; **monotone** means $\{0\}$-monotone. $\omega : V \times \Omega \to V'$ is [P-]monotone iff each ω_A is, $A \in \Omega$. If $P \subset V = \text{PASZ}$, then P is **strongly subdivisible** iff $P \subset V_+$ and to each $\varepsilon \in P$ there is $\delta \in P$ with $2\delta \leq \varepsilon$.

DEFINITION 1. S or (S, Y, d) is **semimetric** iff S is an abelian semigroup with 0, Y a PASZ, $d : S \times S \to Y$ with $d(0, 0) = 0$, $d(u + w, v + w) \leq d(u, v) \leq d(u, w) + d(w, v)$ for $u, v, w \in S$,

(5) $$\mathbf{dd}(\boldsymbol{u}, \boldsymbol{v}) \leq \boldsymbol{y} \quad \text{means} \quad d(u, v) \leq y \quad \text{and} \quad d(v, u) \leq y.$$

If $f_i : X \to S$, then $d(f_1, f_2)(x) := d(f_1(x), f_2(x))$, $x \in X$.

Example 1. Any ASZ S can be made semimetric via the Minkowski functionals ($\inf \phi := \infty$)

(6) $$d_M(a, b) := \inf \left\{ n/m : m, n \text{ natural}, \quad ma \in mb + \sum_1^n M \right\}.$$

If $V \subset P(S)$ with $V \neq \phi$, $0 \in U$ if $U \in V$, $H := [0, \infty]$ and H^V with usual $+, \leq$, $d : S \times S \to H^V$ defined by $d(a, b)(U) := d_U(a, b)$, then (S, H^V, d) is semimetric with $0 = d(a, a) \leq d(a, b)$. See example 9, 10; [14]; [22], [29].

Our first set of **assumptions** is

(7) Ω semiring in X, (S, Y, d) and (S', Y', d') semimetric, P strongly subdivisible $\subset Y'_+$, $W = S$ or Y, with $W' = S'$ or Y' and $\Psi = \Phi$ or ω correspondingly; $\omega : Y \times \Omega \to Y'$ and $\Phi : S \times \Omega \to S'$ biadditive, ω P-monotone, with $d'(\Phi(a, A), \Phi(b, A)) \leq \omega(d(a, b), A)$ for $A \in \Omega$, $a, b \in S$.

LEMMA 2. *Assume* (7). *Then* $(T(\Omega, S), T(\Omega, Y), d)$ *is semimetric*;

(8) $$d'\left(\int h d\Phi, \int k d\Phi\right) \leq \int d(h, k)\, d\omega =: \mathbf{d}_\omega(\boldsymbol{h}, \boldsymbol{k}) \quad \textit{if} \quad h, k \in T(\Omega, S).$$

With $\omega(\beta) := \int \beta d\omega$ *for* $\beta \in T(\Omega, Y)$, ω *is* P-monotone *in* $T(\Omega, Y)$.

So if $\omega : Y \times \Omega \to Y'$ is monotone, $(T(\Omega, S), Y', d_\omega)$ is semimetric. The proof is straightforward, with Lemma 1, Definition 1, and

$$(9) \qquad d'(u_1 + \cdots + u_n, v_1 + \cdots + v_n) \leq d'(u_1, v_1) + \cdots + d'(u_n, v_n) \quad \text{if} \quad u_m, v_m \in S'.$$

2. Proper Riemannian integration

DEFINITION 2. If X, Ω, S, Y, d, W are as in (7), $\omega : Y \times \Omega \to Z = \text{PASZ}$ is biadditive, $P \subset Z_+$, $f \in W^X$, then $f \in \mathbf{R}_0^1(\omega, \mathbf{P}, \mathbf{W})$ iff if f is **proper Riemannian ω-P-integrable** iff to each $\varepsilon \in P$ there are $h \in T(\Omega, W)$ and $\beta \in T(\Omega, Y)$ with (see (5), (4), (1), Lemma 1)

$$(10) \qquad dd(f, h) \leq \beta$$

and, for each $V \in \boldsymbol{r\Omega} := $ ring generated by Ω,

$$(11) \qquad \omega(\beta V) := \int \beta V \, d\omega \leq \varepsilon .$$

$R_0^1(\omega, P, Y)$ is defined if only $X, \Omega, Y, Z, \omega, P$ are given, with $S = \{0\}$, $d \equiv 0$; this case $W = Y = \text{PASZ}$ can be subsumed under the semimetric case using $d(u, v) := \{y \in Y : dd(u, v) \leq y\}$ (see [12], p. 9).

If P is strongly subdivisible $\subset Z_+$, then $f \in R_0^1(\omega, P, W)$ iff to each $\varepsilon \in P$ there is $\beta = \sum_1^n t_m A_m \in T(\Omega, Y)$ with (11) and, for $k = 0, 1, \cdots, n$

$$(12) \qquad d(f_0(x), f_0(y)) \leq \beta_0(x) \qquad \text{if} \quad x, y \in A_k,$$

where $A_0 := \{\infty\} \cup (X - (A_1 \cup \cdots \cup A_n))$, $f_0(\infty) := 0 =: \beta_0(\infty)$. Thus for W = Banach space, Definition 2 is an absolute version of Grave's definition [10]; both coincide if $\dim W < \infty$.

If one has (7) and Y satisfies

$$(13) \qquad a, c \in Y \quad , \quad a \leq a + 2c \qquad \text{always imply} \quad 0 \leq c,$$

then $f \in R_0^1(\omega, P, W)$ iff f satisfies Definition 2, with (11) replaced by $\omega(\beta) \leq \varepsilon$ (and $Z = Y'$), i.e. the V are superfluous.

DEFINITION 3. Under (7), W′ is called **P-complete** if $(P, \leq_{Y'})$ is a net with

$$(14) \qquad a, b \in W' \quad , \quad dd(a, b) \leq \varepsilon \qquad \text{for each} \quad \varepsilon \in P \qquad \text{imply} \quad a = b,$$

and if each P-P-Cauchy-net in W′ converges to some $a \in W'$. A **P-I-Cauchy-net** F in W′ is a $F : I \to W'$ with I a net such that to each $\varepsilon \in P$ there is $\delta \in I$ with $d(F(i), F(j)) \leq \varepsilon$ if $i, j \leq \delta, \in I$; $\mathbf{F} \to \boldsymbol{a}$ means to each $\varepsilon \in P$ there is $\delta \in I$ with $dd(F(i), a) \leq \varepsilon$ if $i \leq \delta$.

Order-complete Y' are P-complete if (14) holds; the converse is usually false: $Y' = R^2$ with $x \leq y$ iff $x = y$ or $x_i < y_i$, $i = 1, 2$, $P = Y'^{+}$ define a P-complete PASZ, Y' is not order-complete, Y'_{+} is not even a lattice. $P \neq \phi$ if $\{0\} \neq W' =$ P-complete. Usually we need

$$(15) \qquad (7) \text{ holds, with } S' \text{ and } Y' \text{ P-complete}.$$

THEOREM A. *Assume* (7). *Then* $(R_0^1(\omega, W, S), R_0^1(\omega, P, Y), d)$ *is semimetric.* $T(\Omega, W) \subset R_0^1(\omega, P, W)$, $fV \in R_0^1(\omega, P, W)$ *if* $f \in R_0^1(\omega, P, W)$, $V \in r\Omega$; *if* (7) *holds and* W' *is* P-complete, *then there is exactly one* P-continuous $\Psi : R_0^1(\omega, P, W) \to W'$ *with* $\Psi(h) = \int h d\Psi$ *if* $h \in T(\Omega, W)$; *this* Ψ *is additive in* $R_0^1(\omega, P, W)$, *and* P-monotone *for* $\Psi = \omega$; *we write also* $\int \cdots d\Psi$.

Ψ **P-continuous** at $g \in R_0^1(\omega, P, W)$ means to $\varepsilon \in P$ there is $\delta \in P$ such that $d'd'(\Psi(f), \Psi(g)) \leq \varepsilon$ whenever $f \in R_0^1(\omega, P, W)$, $h \in T(\Omega, Y)$, $dd(f, g) \leq h$, $\omega(h) \leq \delta$ (no V). Theorem A can be applied if only X, Ω, Y, Y', ω, P are given, with $S = S' = \{0\}$, $\Phi \equiv 0$.

Proof. The first statement follows with Lemma 2, with pointwise $=, +, \leq$ in $R_0^1 := R_0^1(\omega, P, W)$; especially $d(f, g) \in R_0^1(\omega, P, Y)$ if $f, g \in R_0^1(\omega, P, S)$. $T \subset R_0^1$ since $d(h, h) \leq d(0, 0) = 0$. By Definition 2 and the axiom of choice to $f \in R_0^1$ exist $H : P \to T(\Omega, W)$, $B : P \to T(\Omega, Y)$ with $dd(f, H(i)) \leq B(i)$, $\omega(B(i)V) \leq i$ for $V \in r\Omega$, $i \in P$. $\Psi(H)$ is P-P-Cauchy by Lemma 2, $\Psi(H) \to : \Psi(f) \in W'$. With (14) the lim in W' is unique and $\Psi(f)$ independent of H, B, yielding existence of an additive Ψ. Continuity implies uniqueness since T is 'dense' in R_0^1. For continuity resp. P-Monotonicity of Ψ one shows with Lemma 2: If $f, g \in R_0^1$, $h \in T(\Omega, W)$, $d(f, g) \leq h$, $\varepsilon \in P$, then (see (4))

$$(16) \qquad d'(\Psi(f), \Psi(g)) \leq \omega(h) + \varepsilon. \quad \square$$

COROLLARY. *If besides* (15) *always*

$$(17) \qquad a, b \in Y', \quad a \leq b + \varepsilon \quad \textit{for all } \varepsilon \in P \quad \textit{imply } a \leq b,$$

then $(R_0^1(\omega, P, S), Y', d_\omega)$ *is semimetric,* $T(\Omega, S)$ d_ω-dense *in* R_0^1,

$$(18) \qquad d'\left(\int f d\Phi, \int g d\Phi\right) \leq \int d(f, g)\, d\omega =: d_\omega(f, g),$$

$$f, g \in R_0^1 := R_0^1(\omega, P, S).$$

d_ω**-dense** means: To $\varepsilon \in P$, $f \in R_0^1(\omega, P, W)$ exists $h \in T(\Omega, W)$ with $d_\omega d_\omega(fV, hV) \leq \varepsilon$ for $V \in r\Omega \cup \{X\}$.

Example 2. $X = R^n$, $\Omega = \Omega_n :=$ all $I = I_1 \times \cdots \times I_n$, $I_m = [a_m, a_m + l_m)$, $S = S' = Y = Y' =$ reals R, usual $+, \leq$, $d = d'$, $\omega(t, I) = t\mu_L(I)$,

$\mu_L(I) = l_1 \cdots l_n$, ($\sigma$-) additive in Ω_n = semiring, $\Phi = \omega$, $P = R^+$: $R_0^1(\omega, P, W)$ = exactly all proper Riemann integrable $f: R^n \to R$ in the usual sense, $\int f d\Psi$ = usual Riemann integral $\int_{R^n} f dx$. See example 3, 4, 5.

Example 3. If Y = group in (7), Y′ satisfies (13), then $f \in R_0^1(\omega, P, Y)$ iff to each $\varepsilon \in P$ there are $h, k \in T(\Omega, Y)$ with

$$(19) \qquad h \leq f \leq k \quad \text{and} \quad \int kV\, d\omega \leq \int hV\, d\omega + \varepsilon \qquad \text{for } V \in r\Omega .$$

Y′ is P-complete if Y′ = order-complete ordered vectorspace V [28], P as in (7), net with (17), Theorem A is applicable to Y, ω. For $R_0^1(\omega, P, Y) \subset$ or = Darboux-integrable functions $D_0^1(\omega, Y)$ see [13] (D 23) ff; especially the integral in [4] is subsumed.

If E = Riesz space [21], $\mu: \Omega$ semiring $\to R$ additive (no finite variation), $Y = Y' = E$, $\omega(a, A) = \mu(A)\, a$, P as in (7), then besides (12)

$$(20) \qquad R_0^1(\mu, P, E) := R_0^1(\omega, P, E_i) = \{f \in E^X : f \text{ satisfies (19)}, \ \varepsilon \in P\}$$

for $i = 1, 2, 3$, this is a Riesz space with pointwise $f \wedge g, |f|$; here $E_1 = (E, E_+, |a - b|)$, $E_2 = (E, E, a - b)$, $E_3 := E$ as PASZ. Examples: $E = l^p$, $1 \leq p \leq \infty$, $\varepsilon \in P$ iff all $\varepsilon_m > 0$; if $E = R^n$, also $Y = R$, $\omega = \mu(A) \sum_1^n a_k$ or $E_4 = (R^n, R_+, d)$ with invariant metric d with usual topology, $E_5 = R^n$ as PASZ with $a \leq b$ iff $a = b$ or $a_k < b_k$, $P = Y^+$, are possible in

$$(20) = R_0^1 \times \cdots \times R_0^1 \quad , \quad R_0^1 := \text{any such for } n = 1 ; \ E_5 \text{ is not Riesz}, \ n > 1 .$$

Similarly for $\mu: \Omega \to V$ or $E, f \in R^X$, or μ, f F-valued, F ordered field. See example 4.

Example 4. $v: \Omega$ semiring $\to R_+$ additive, B_i (closed convex cones $\subset$) Banach spaces, $\mu: \Omega \to B_3$ additive, $\|\mu(A)\| \leq v(A)$, $A \in \Omega$ (p.e. v = variation $|\mu|$). $\Phi_0: B_1 \times B_3 \to B_2$ bilinear and continuous. If $S = B_1$, $S' = B_2$, $Y = Y' = R_+$, $\Phi(a, A) = \Phi_0(a, \mu(A))$, $\omega(t, A) = \|\Phi_0\|\, tv(A)$, $P = R^+$, one has (13), (15), (17), (22), Theorems A-G hold: With $\|f\| := \int |f|\, dv$, $|f|(x) := \|f(x)\|_1$, $R_0^1 := R_0^1(v, B_1) := R_0^1(\omega_0, R^+, B_1)$ is a prenormed linear space, $f \in R_0^1 \Rightarrow |f| \in R_0^1(v, R)$, $\int \Phi_0(f, d\mu) := \int f d\Phi$ and $\int \varphi dv := \int \varphi d\omega_0$ are well defined and linear for $f \in R_0^1$, $\varphi \in R_0^1(v, R)$, $\omega_0(t, A) := tv(A)$,

$$(21) \qquad \left\| \int \Phi_0(f, d\mu) \right\| \leq \|\Phi_0\| \cdot \|f\| \qquad f \in R_0^1 .$$

By (12), (13), $f \in R_0^1$ iff to each $\varepsilon > 0$ there are $A_1, \cdots, A_n \in \Omega$ with $\sum_1^n v(A_k) \sup_{A_k} \| f(x) - f(y) \| < \varepsilon$ and $f = 0$ on $X - \bigcup_1^n A_k$ iff to each $\varepsilon > 0$ there are $h, k \in T(\Omega, B_1)$ with $|h| \leq |f| \leq |k|$, $|f - h| \leq |k - h|$, $\int |k - h| \, dv < \varepsilon$, a B-analogue to (19). If $\Omega = \Omega_n$, $v = \mu_L$ of example 2, *then* $f \in R_0^1(\mu_L, B)$ *iff* f *is bounded, has bounded support, and is Lebesgue-a.e. continuous*; this is false for the Graves $-\int$ [25], [26].

Similar results hold for E_i Riesz spaces, $i = 1, 2, 3$, $P \subset E_3^+$, $v : \Omega \to E_{3+}$, with 'Φ_0 continuous' replaced by $\Phi_0 \geq 0$, i.e. $\Phi_0(a, b) \geq 0$ if $a \in E_{1+}$, $b \in E_{3+}$, the norms replaced by $| \ |$ of E_i, $\omega(t, A) = \Phi_0(t, v(A))$, $\omega(|f|)$ on the right of (21).

Example 5. $S = S' =$ field F with real valuation φ [15], complete, $\mu : \Omega \to F$ and $v : \Omega \to R$ additive with $\varphi(\mu(A)) \leq v(A)$; as in example 4, $R_0^1(v, F)$ is F-linear and (Theorem B) a ring, $\int f d\mu$ F-linear, $\varphi\left(\left(\int f d\mu\right)\right) \leq$ $\leq \int \varphi(f) \, dv = \| f \|$, a prenorm. Special cases: $F = p$-adic numbers, or the integrals in [24].

THEOREM B. *Assume* (7), (S_i, Y, d_i) *semimetric*, $f_i \in R_0^1(\omega, P, S_i)$, $M : S_1 \to S$, $T : S_1 \times S_2 \to S$, $M(0) = T(0, 0) = 0$, M *and* T *Lipschitzian. Then* $M(f_1)$ *and* $T(f_1, f_2) \in R_0^1(\omega, P, S)$.

M resp. T (locally) **Lipschitzian** means there is a natural n_0 (for T depending on f_1) such that to each finite $Q \subset S_2$ there is a natural n_Q with $d(M(u), M(v)) \leq n_0 \, d_1(u, v)$ resp. $d(T(f_1(x), y), T(f_1(x), z) \leq n_0 \, d_2(y, z)$ and $d(T(u, q), T(v, q)) \leq n_Q \, d_1(u, v)$ if $u, v \in S_1$, $y, z \in S_2$, $x \in X$, $q \in Q$. For a proof see [12] p. 11.

If M is additive and S', S_1' are P-complete, then

$$\int M(f) \, d\Phi = M'\left(\int f d\Phi_1\right) \quad \text{with suitable } M', \Phi_1 .$$

Thus under the assumptions of example 4, $\Phi_0(f_1, f_3) \in R_0^1(v, B_2)$ if $f_i \in R_0^1(v, B_i)$; especially $\varphi \cdot f \in R_0^1(v, B)$ if $f \in R_0^1(v, B)$, $\varphi \in R_0^1(v, \Sigma)$, $\Sigma =$ reals R. resp. complex numbers C; $R_0^1(v, \Sigma)$ is an algebra.

DEFINITION 4. Assuming (7) and $M \subset X$, then $M \in J_0(\omega, P)$ [resp. $\in N_0(\omega, P)$] iff M is a **proper Jordan ω-P-set** [resp. **proper Jordan ω-P-nulset**] iff to each $\varepsilon \in P$, $y \in Y^+$ there are $U, V \in r\Omega$ with $U \subset M \subset U \cup V$ and $\omega(yV) \leq \varepsilon$ [and $U = \phi$].

For $X = R^n, \cdots$ as in example 2, these are exactly the usual (proper) Jordan [nul]sets.

THEOREM C. *Assume* (7) *and* $G \subset X$. *Then* $J_0(\omega, P)$ *is a ring containing* $N_0(\omega, P)$ *as an ideal*; (a) $G \in J_0(\omega, P)$ *iff* (b) $yG \in R_0^1(\omega, P, Y)$ *for* $y \in Y^+$. *If also* $M \in J_0(\omega, P)$, $f \in R_0^1(\omega, P, W)$, $a \in W$ *and*

$$(22) \qquad \text{to } u \in Y \text{ always exists } y \in Y_+ \quad \text{with} \quad 0 \leq u + y,$$

then fM, $aM \in R_0^1(\omega, P, W)$ *and* (a), (b), (c) $:=$ (b) *for* $y \in Y_+$, (d) $:=$ (b) *for* $y \in Y$ *are equivalent*; *if additionally* $Y^+ \subset d(S, S)$, *then* (a) — (e) *are equivalent*, (e) $:=$ (b) *for* $y \in S$.

Proof. $J_0 =$ ring follows with the P-monotonicity of ω, Lemma 2, since $Yr\Omega \subset T(\Omega, Y)$; N_0 is even an ideal in $P(X)$.

$a \Rightarrow b$ with $h = yU$, $\beta = yV$.

$b \Rightarrow a$: With δ, h, β, B_m as below for $f = yG$, let $U :=$ union of $B_m \subset G$, $V :=$ union of B_m meeting G, $\not\subset G$. Then $U \subset G \subset U \cup V$ if $y > 0$; in V, $h \leq \beta$, $y \leq h + \beta$; so $yV \leq 2\beta V$, $\omega(yV) \leq 2\delta + \delta \leq \varepsilon$. If $Y^+ = \phi$, $J_0 = P(X)$.

For $fM \in R_0^1$, to $\varepsilon \in P$ choose $\delta \in P$, h, β, b_m, b'_m, B_m as in (2), $k = \beta$, with $4\delta \leq \varepsilon$, $dd(f, h) \leq \beta$, $\omega(\beta Q) \leq \delta$, then u_m, v_m, x'_m, $x''_m \in Y_+$ with $0 \leq b'_m + v_m$, $0 \leq b_m + u_m$ and $y_m := b_m + 2u_m + v_m$ if $W = Y$, else $0 \leq d(0, b_m) + x'_m =: u'_m$, $0 \leq d(b_m, 0) + x''_m =: u''_m$, $y_m := u'_m + u''_m + v_m$; $y := \sum_1^n y_m$, $y \in Y_+$. If $y > 0$ there are disjoint U, $V \in r\Omega$ with $U \subset M \subset$ $\subset U \cup V =: U'$, $\omega(yVQ) \leq 2\delta$. With $\beta' := \beta U' + yV$, $\omega(\beta' Q) \leq 3\delta \leq \varepsilon$ and $dd(hU, fM) \leq \beta'$, for $W = S$ with $d(0, f) \leq d(0, h) + d(h, f) \leq$ $\leq y + \beta$ etc. This holds also for $y \leq 0$, with $U := \phi$, $V := B_1 \cup \cdots \cup B_n$, since now $\beta' \geq 0$. So $fM \in R_0^1$, even if $Y^+ = \phi$. For $aM \in R_0^1$, if $Y^+ \neq \phi$ choose $Q \in r\Omega$ with $M \subset Q$; since $aQ \in R_0^1$, $aM = aQM \in R_0^1$; if $Y^+ = \phi$, $R_0^1 = W^X$, for $0 \leq Y \leq 0$ by (22).

Now $a \Rightarrow e$, $a \Rightarrow d \Rightarrow c \Rightarrow b$. $e \Rightarrow c$: $d(a, b) G = d(aG, bG) \in R_0^1$ by Theorem A.

Instead of complete, the R_0^1 are at least 'closed' by Definition 2:

THEOREM D. *Under* (7) *resp.* (15), $f \in R_0^1(\omega, P, W)$ *iff to each* $\varepsilon \in P$ *there are* $g \in R_0^1(\omega, P, W)$, $\varphi \in T(\Omega, Y)$ *resp.* $\in R_0^1(\omega, P, Y)$ *with* $dd(f, g) \leq \varphi$ *and* $\omega(\varphi V) \leq \varepsilon$ *for all* $V \in r\Omega$.

The extension process step-functions $\rightarrow$ proper Riemann integrable functions is iteration complete: $R_0^1(\omega, P, W) = R_0^1(\omega^*, P, W)$ with $\omega^* : Y \times J_0(\omega, P) \rightarrow Y'$, $\omega^*(y, M) = \int yM \, d\omega$, with (15).

Special cases of Definition 2 are the integrals in [1] sec. 8, [17] p. 356, [19] p. 170, [3], [20], [27], [31], [32]. See [13], (110).

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DEFINITION 5. If (7) holds, $f, g \in W^X$, $Q \subset X$, $F : I \to W^X$ with I net, then $F \to f$ **ω-P-locally** or $\mathbf{F \to f(\omega)}$ iff to each $\varepsilon \in P$, $A \in \Omega$, $y \in Y$ there are $j \in I$, $B : I \to T(\Omega, Y)$, $M : I \to r\Omega$ with, for $V \in r\Omega$, $i \leq j$, $i \in I$:

$$\text{(23)} \qquad dd(f, F(i)) \leq B(i) \quad \text{on} \quad A - M(i)$$

$$\text{(24)} \qquad \int B(i) V d\omega \leq \varepsilon \quad , \quad \int yM(i) V d\omega \leq \varepsilon ;$$

$\mathbf{f = g(\omega)}$ iff $F \to g(\omega)$ with $F = f$ on some net; $f \in \mathbf{n}(\omega, \mathbf{P}, \mathbf{W})$ iff f is a **ω-P-nulfunction** iff $f = 0(\omega)$; $Q \in \mathbf{N}(\omega, \mathbf{P})$ iff Q is a **ω-P-nulset** iff $yQ \in n(\omega, P, Y)$ for each $y \in Y$.

DEFINITION 6. If (7) holds and $f \in W^X$, then $f \in \mathbf{R}^1(\omega, \mathbf{P}, \mathbf{W})$ iff f is **Riemann ω-P-integrable** iff there is $H : I \to T(\Omega, W)$ with $I = P$ and $\mathbf{H \Rightarrow f(\omega)}$, meaning $H \to f(\omega)$ and H is a **d_ω-P-Cauchy-net**, i.e. to each $\varepsilon \in P$ is $\delta \in I$ with, for $V \in r\Omega$, $i, j \in I$, $i \leq \delta$, $j \leq \delta$, $W = S$ resp. Y

$$\text{(25)} \qquad \int d(H(i), H(j)) V d\omega \leq \varepsilon \quad \text{resp.} \quad \int H(i) V d\omega \leq \int H(j) V d\omega + \varepsilon .$$

Simplifications: Under (13) and (22), the V's can be dropped in (24); also in (25), case S, if $d \geq 0$; equivalently $M \in J_0$. If $X \in r\Omega$, $A = X$ suffices in (23).

THEOREM E. *If* (7) *and* (22) *hold, then Theorem* A *and its corollary remain true, if everywhere* R^1_0 *is replaced by* R^1. $R^1_0(\omega, P, W) \subset R^1(\omega, P, W)$, *the integrals on* R^1 *extend those of* R^1_0.

$\Psi : R^1(\omega, P, W) \to W'$ **continuous** means if $g \in R^1(\omega, P, W)$, $H : I \to T(\Omega, W)$, $H \Rightarrow g(\omega)$, then $\Psi(H) \to \Psi(g)$: by (18) and Lemma 3, in Theorem E this is under (15) equivalent with ε-δ-continuity with respect to d_ω; see also Theorem G.

Proof. If $F_k \to f_k(\omega)$, then $F_1 + F_2 \to f_1 + f_2(\omega)$, $d(F_1, F_2) \to d(f_1, f_2)(\omega)$, similarly for $\Rightarrow$, yielding the first statement of Theorem A for R^1, under pointwise $=, +, 0, \leq, d$. $R^1_0 \subset R^1$ with $M = \phi$.

LEMMA 3. *Assume* (7), $H : I \to T(\Omega, W)$, $K : J \to T(\Omega, W)$, $H \Rightarrow f(\omega)$, $K \Rightarrow g(\omega)$, $f, g \in W^X$, *and* $f = g$ *if* $W = S$; *if* $W = Y$, $f \leq g$ *and* (22); $\varepsilon \in P$. *Then there are* $i_\varepsilon \in I$, $j_\varepsilon \in J$ *with, for* $V \in r\Omega$, $i \leq i_\varepsilon$, $j \leq j_\varepsilon$: $\omega(d(H(i), K(j))V) \leq \varepsilon$ *resp.* $\omega(H(i)V) \leq \omega(K(j)V) + \varepsilon$.

Proof. Assume $6\delta \leq \varepsilon$, $d_\omega(H(i)V, H(i')V) \leq \delta$, $d_\omega(K(j)V, K(j')V) \leq \delta$ if $i, i' \leq i_\delta$, $j, j' \leq j_\delta$, $h := H(i_\delta)$, $k := K(j_\delta)$ as in (2). Choose $\beta \in P$, $y_m \in Y$ with $4n\beta \leq \delta$, $y_m := d(b_m, b'_m)$ resp. $b_m \leq b'_m + y_m$.

There are $i_0 \leq i_\delta$, independent of m, and $C_m \in T(\Omega, W)$, $M_m \in r\Omega$, $\subset B_m$, with $d(H(i_0), f) \leq C_m$ on $B_m - M_m$, $\omega(C_m V) \leq \beta$, $\omega(y_m M_m V) \leq \beta$; similarly one has j_0, D_m, P_m with $d(g, K(j_0)) \leq D_m$ on $B_m - P_m$, $\omega(D_m V) \leq \beta$, $\omega(y_m P_m V) \leq \beta$. If $Q_m := M_m \cup P_m$, $U := \bigcup_1^n (B_m - Q_m)$, $W = S$ then $d(h, k) \leq d(h, H(i_0))$, $U + \sum_1^n (C_m + D_m) U + d(K(j_0), k) U + \sum_1^n y_m Q_m$, with $\omega(y_m Q_m V) = \omega(y_m M_m V) + \omega(y_m P_m (P_m - M_m) V) \leq 2\beta$. So if $i \leq i_\delta$, $j \leq j_\delta$, interpolating hV, kV, $d_\omega(H(i) V, K(j) V) \leq \delta + (\delta + n(\beta + \beta) + \delta + n 2\beta) + \delta + \delta \leq 6\delta \leq \varepsilon$. Similarly if $W = Y$, $f \leq g$. □

With this and (8), $\lim \int H d\Psi$ exists $= \lim \int K d\Psi =: \Psi(f)$ if $f = g$. $\Psi : R^1 \to W'$ is well defined $(I = P)$, additive, continuous (any net I), unique etc. If $H \Rightarrow f$, $K \Rightarrow g(\omega)$, $H, K \subset T(\Omega, W)$, i' fixed $\in I$, $h(i) := H(i')$, $V \in r\Omega$ or $V = X$, then $dd(H, h) V \Rightarrow dd(f, H(i')) V(\omega)$ by the above; continuity and P-Monotonicity of ω, (25), (4), (8) yield under (15) ($\to (d_\omega)$: p. 13, line 2; $W = S$ for the last $\to$) *uniformly in* $V \in r\Omega \cup \{X\}$,

$$(26) \quad HV \to fV(d_\omega), \quad \int HV\, d\Psi \to \int fV\, d\Psi, \quad d_\omega(HV, KV) \to d_\omega(fV, gV)$$

(if $W = S$ and $d(f, f) \geq 0$, even $d_\omega d_\omega(HV, fV) \to 0$ in Y', V-uniformly). Especially T is d_ω-dense in R^1 as after (18), (26) and (8) also yield (18) with $+\varepsilon$; so if (17) holds, one has (18), ω is monotone, d_ω a semimetric on $R^1(\omega, P, S)$. □

Riemann integrable functions which are not proper Riemann integrable always have singularities:

COROLLARY. *If* (7) *holds,* $P \neq \phi$, $f \in W^X$, *then* (a) $f \in R_0^1(\omega, P, W)$, (b) $f \in R^1(\omega, P, W)$ *and* f *is* Ω*-bounded and* (c) *there exists* $H : I$ net $\to T(\Omega, T)$ *with* $H \to f(\omega)$ *and* f *is* Ω*-bounded are equivalent.*

$g \in W^X$ is **Ω-bounded** iff there is $h \in T(\Omega, W)$, $\beta \in T(\Omega, Y)$ with $dd(g, h) \leq \beta$. Under (22) g is Ω-bounded iff there are $u \in Y_+$, $V \in r\Omega$ with $dd(f, 0) \leq uV$.

Proof. $c \Rightarrow a$: If $dd(fA, aA) \leq uA$, B, M as in Definition 5, for some i_ε $dd(fA, H(A - M) + aAM) \leq B(A - M) + uAM =: k$ on X and $\omega(kV) \leq \varepsilon$, so $fA \in R_0^1 := R_0^1(\omega, P, W)$. If $d(f, h) \leq \beta$, h, β as in (2), then $fB_m \in R_0^1$. $f_0 := fB_1 + \cdots + fB_n \in R_0^1$ by Theorem A. $dd(f, f_0) \leq 0$, so $f \in R_0^1$. □

By Definition 5, $= (\omega)$ is a local concept, so $f = g(\omega)$ or $f =$ nulfunction or $Q =$ nulset iff $fA = gA(\omega)$ or $fA =$ nulfunction or $QA =$ nulset for each $A \in \Omega$; then this holds also for $A \in r\Omega$.

To collect some further properties, for Theorem F we abbreviate: $n = n(\mathrm{W}) = n(\omega, \mathrm{P}, \mathrm{W})$, $\mathrm{N} = \mathrm{N}(\omega, \mathrm{P})$, $\mathrm{R}^1 = \mathrm{R}^1(\omega, \mathrm{P}, \mathrm{W})$, and assume

$$f, g, k \in \mathrm{W}^{\mathrm{X}}, \quad \mathrm{Q} \subset \mathrm{X}, \quad p_i \in n, \quad q_i \in n(\mathrm{Y}), \quad \text{and} \tag{27}$$
$$\text{if} \quad y \in \mathrm{Y}, \quad 0 \leq 2y, \quad \text{then} \quad 0 \leq y.$$

THEOREM F. *Assume* (15), (22), (27), $f \in \mathrm{R}^1$. *Then* $n \subset \mathrm{R}^1$, $(n(\mathrm{S}), n(\mathrm{Y}), \mathrm{d})$ *is semimetric,* N *an ideal in* P(X), $= (\omega)$ *an equivalence relation in* W^{X}; *if* $g = f(\omega)$, *then* $g \in \mathrm{R}^1$, $\int g \mathrm{d}\Psi = \int f \mathrm{d}\Psi$; $\int p_1 \mathrm{d}\Psi = 0$; *if* $\mathrm{d}(g + p_1, f + p_2) \leq q_1$ *and* $\mathrm{d}(f + p_3, g + p_4) \leq q_2$, *then* $f = g(\omega)$; *if* $0 \leq h \in \mathrm{Y}^{\mathrm{X}}$, *then* $h \in n$ *iff* $h \in \mathrm{R}^1$ *and* $\int h \mathrm{d}\omega = 0$; $\mathrm{Q} \in \mathrm{N}$ *iff* $\mathrm{QA} \in \mathrm{N}_0(\omega, \mathrm{P})$ *iff* $y\mathrm{Q} \in \mathrm{R}^1$ *and* $\int y\mathrm{Q}\, \mathrm{d}\omega = 0$ *for* $y \in \mathrm{Y}^+$, $\mathrm{A} \in \Omega$.

So arbitrary (unbounded) changes on nulsets do not change $= (\omega)$, $\in \mathrm{R}^1$, $\in n$, the integrals, $\in \mathrm{N}$; $\mathrm{WN} \subset n$; $f = g + p_1$ or $g = f + p_1$ implies $f = g(\omega)$, $g \in \mathrm{R}^1$; if $\mathrm{d} \geq 0$, $g = k(\omega)$ iff $\mathrm{dd}(g, k) \in n(\mathrm{Y})$.

Proofs for all this, generalizations and further results can be found in [12], p. 21, 22, 27.

If X, ··· are as in example 2, then R^1 contains all $f : \mathrm{R}^n \to \mathrm{R}$ which together with $|f|$ are improper Riemann integrable in the usual sense; however, here one can also construct $f \in \mathrm{R}^1$ with even uncountably many singularities or nulfunctions p with $\{p \neq 0\}$ not a nulset.

Theorems E-G can also be applied in the situations of examples 3, 4, 5, or to S = Frechet-space in the sense of Banach: P.e. the integrals are linear, one has (21) for $f \in \mathrm{R}^1(v, \mathrm{B}_1) := \mathrm{R}^1(\omega_0, \mathrm{R}^+, \mathrm{B}_1)$, a linear prenormed space as in example 4 (or 5 or 7); $\mathrm{H} \to f(\omega)$ iff to each real $\varepsilon > 0$, $\mathrm{A} \in \Omega$ there is $\mathrm{M} \in r\Omega^{\mathrm{I}}$ and δ with $|f - \mathrm{H}(i)| \leq \varepsilon$ on $\mathrm{A} - \mathrm{M}(i)$ and $v(\mathrm{M}(i)) \leq \varepsilon$ if $i \leq \delta$; if $\mathrm{H} \Rightarrow f(\omega)$, there is δ independent of A.

The $\mathrm{R}^1(v, \mathrm{B}_1)$ always contains L(S) of Dunford-Schwartz [8] p. 112; only if $\mathrm{X} \in r\Omega$ one has equality. By example 7, $\mathrm{R}^1(v, \mathrm{R})$ coincides with Loomis' improper integrable functions [19].

Example 6. X infinite, $\Omega = \{\mathrm{M} \text{ or } \mathrm{X} - \mathrm{M} \text{ finite}\}$, $\mu(\mathrm{M}) = 0$ resp. 1, $\mathrm{Y} = \mathrm{Y}'$, $\mathrm{S} = \mathrm{S}'$, P, W as in (15), $\omega(y, \mathrm{M}) = \mu(\mathrm{M})\, y$, $\Phi(a, \mathrm{M}) = \mu(\mathrm{M})\, a$; then $\mathrm{R}^1(\omega, \mathrm{P}, \mathrm{W}) = \mathrm{R}^1_0(\omega, \mathrm{P}, \mathrm{W}) = \{f \in \mathrm{W}^{\mathrm{X}} : \text{P-lim } f \text{ exists}, \int f \mathrm{d}\Psi = \text{P-lim } f$ (as in Definition 3, for the filter $\{f(\mathrm{M}) : \mathrm{X} - \mathrm{M} \text{ finite}\}$ (or compact)).

Example 7. If X, ···, v, μ, ω, Φ, P are as in example 4, $\|a\|^{-1} := 0$ if $a = 0$, $t \cap a := \min(t, \|a\|) \cdot a \cdot \|a\|^{-1}$ for $a \in \mathrm{B}_1$, $t \in \mathrm{R}_+$, with corresponding pointwise $\varphi \cap f$ if $f \in \mathrm{W}^{\mathrm{X}}$, $\varphi \in \mathrm{R}^{\mathrm{X}}_+$, $\mathrm{W} = \mathrm{B}_1$ or R, *then* $f \in \mathrm{R}^1(v, \mathrm{W})$

iff $|||f||| := \sup \left\{ \int h dv : 0 \leq h \leq |f| \,,\, h \in T(\Omega, R) \right\} < \infty$ and $h \cap f \in R_0^1(v, B_1)$ *for each* $h \in T(\Omega, R_+)$; $|||f||| = \int f dv$ *follows.* The proof uses $\| t \cap a - t \cap b \| \leq 2 \min(t, \| a - b \|)$; $B_1 = R$ yields Loomis' definition [19]; [3], [33] treat special cases.

If $g \in R^1(v, W)$, $t \in R_+$, then $t \cap g \in R^1(v, W)$ and $||| tV \cap g - gV ||| \to 0$ as $t \to \infty$, $||| tV \cap g ||| \to 0$ as $t \to 0+$, uniformly in $V \in r\Omega \cup \{X\}$; this holds also for W = Riesz space E_1 of ex. 3/4, with $t \cap a := (t \wedge a) \vee (-t)$, $t \in E_+$; then $R^1(\omega, P, E)$ is a Riesz space as in example 3 ($\Phi_0(t, a) \to 0$ is needed as $t \to 0+$, $a \in E_3$). See [13].

Example 8. $Y = Y' = [0, \infty]$, B Banach space, $S = S' = \{U \subset B : \phi \neq U$ closed convex$\}$ with Hausdorff 'metric' d, $U + V :=$ closure $\{u + v : u \in U, v \in V\}$, v, Ω as in example 4, $\omega := tv(A)$ with $\infty \cdot 0 := 0$, $\Phi := v(A) \cdot U := \{v(A) u : u \in U\}$, $P := (0, \infty)$. Then (13), (15), (22), (27) hold, all our results are applicable; p.e. if $f \in F \in R_0^1(v, S)$, $f \in R_0^1(v, B)$, then $\int f dv \in \int F d\Phi$. Similarly one can use additive $\mu : \Omega \to S$ as above with $d(\mu(A), 0) \leq v(A)$; closed can be replaced by closed bounded or compact, then $d < \infty$; or S as PASZ, with the same R^1 ($P = \{\{u \in B : \| u \| \leq \varepsilon\} : \varepsilon > 0\}$). Thus analogues to, respective extensions of, the integration theories in [2], [6] follow.

Example 9. $S' = S =$ complete T_1-topological abelian group, p.e. locally compact, $V := \{0 \in U = -U$ open $\subset S\}$; if to $U = U_1 \in V$ one chooses $U_k \in V$ with $U_k + U_k + U_k \subset U_{k-1}$, $U_0 = S$, $q_U(a) := \inf\{2^{-k} : a \in U_k\}$, $p(a)(U) := \inf \sum_1^n q_U(a_m - a_{m-1})$ with inf over n and $a_m \in S$ with $a_0 = 0$, $a_n = a$, $Y' = Y := H^V$ as in example 1, $0 \leq d(a, b) = d(b, a) := p(b - a)$, $\varepsilon \in P$ iff $0 < \varepsilon(U)$ for $U \in V$ and $\varepsilon(U) = \infty$ except for finitely many U, then (S, Y, d) is semimetric P-complete (Birkhoff; see [18] p. 46/47). If $\mu : \Omega$ semiring $\to \{0, 1, 2, \cdots\}$ is additive, $\Phi(a, A) = \mu(A) a$, $\omega(t, A) = \mu(A) t$, then (13), (15), (17), (22), (27) hold, all our theorems are applicable. Another possibility: $Y' = Y = \{\phi \neq F$ closed $\subset S\}$, $d(a, b) := \{b - a\}$, $P = \{\bar{U} : U \in V\}$, $\omega(F, A) =$ closure of $\sum_1^{\mu(A)} F$. See example 10, 11; [14], [22], [29].

Example 10. $S' = S = L =$ convex closed cone $\subset$ locally convex topological space, V = neighborhood basis for 0 of absolutely convex open sets, $Y' = Y = H^V$ and d as in example 1, $q(u) := d(u, 0)$, $\mu : \Omega$ semiring $\to R_+$ additive, Φ, ω, P as in example 9; then (7), (13), (17), (22), (27) hold, Y is P-complete, all our theorems except A/E, case W = S P-complete, are applicable, especially $R^1(\mu, L_V)$ is a cone or linear, $L_V = (L, H^V, d)$. Similar (lager) R^1 are obtained if $V = \{((\text{bounded})\text{ convex})\text{ open } U \ni 0\}$. If S = B-space B, then $R^1(\mu, L_j)$ coincide, $j = V$, $\{0 \in U$ open ball$\}$, $\|$ with

$L_{\|} = (B, R_+, \| \|)$. If L is complete, p.e. Schwartz's $\mathscr{D}$ or $\mathscr{D}'$, then L_V is P-complete, all theorems apply.

When almost everywhere convergence is replaced by convergence ω-P-locally, then the various convergence theorems for the Lebesgue-Bochner integral hold also in the finitely additive case and R^1, p.e. (for extensions, proofs see [12] p. 28).

THEOREM G. *Assume* (7), Y′ P-complete, (22), *and* $0 \leq d$ *on* $S \times S$, *or* d *symmetric*; $f \in S^X$, *I net*, $F : I \to R^1(\omega, P, S)$. *Then* $F \Rightarrow f(\omega)$ *iff* $f \in R^1(\omega, P, S)$ *and* $F \to f(d_\omega)$.

So if S′ is P-complete, also $\int FV \, d\Phi \to \int fV \, d\Phi$, uniformly in $V \in r\Omega \cup \{X\}$.

As in Definition 6, $\mathbf{F} \Rightarrow \boldsymbol{f}(\omega)$ means $F \to f(\omega)$ and F is a d_ω-P-Cauchy-net as in (25); $\mathbf{F} \to \boldsymbol{f}(\mathbf{d}_\omega)$ means to each $\varepsilon \in P$ there is $j \in I$ with $d_\omega\, d_\omega(F(i)V, fV) \leq \varepsilon$ if $i \leq j$, $V \in r\Omega$, d_ω of (18).

As after Theorem D, the extension $T(\Omega, W) \to R^1(\omega, P, W)$ is iteration complete; there are analogues to Theorems B, C.

If **integrability over** $M \subset X$ **of** $f : D \to W$ is defined by $f_0 M \in R^1(\omega, P, W)$, $f_0 := 0$ on $X - D$, **measurability** iff there is $F : I \to R^1(\omega, P, W)$ with $F \to f_0 M(\omega)$, our theorems hold also; however not even in example 2 all $\{f > t\}$ are Jordan sets, $0 < t$, $f \in R^1_{(0)}(\mu_L, R)$ ([13]) (D27)). For Fubini theorems see [9].

Let $\mathbf{R}^1_\sigma$ contain all $f \in R^1$ to each of which exist $A_n \in \Omega$ with $f = f \bigcup_1^\infty A_n$.
If there are $\varepsilon_1, \varepsilon_2, \cdots, \in P$ with $\varepsilon_m \to 0$ (Definition 3), then to each $f \in R^1$ there is $g \in R^1_\sigma$ with $f = g(\omega)$, so that $\mathscr{R}^1$ and $\mathscr{R}^1_\sigma$ coincide, $\mathscr{R}^1$ denoting R^1 mod ω.

If the Lebesgue-Bochner integral is defined (example 4, μ, v σ-add.)

$$R^1_\sigma(v, B_1) \subset L^1_1(v, B_1) \tag{28}$$

with coinciding integral (seminorm) (one can extend L^1 so that $R^1 \subset L^1$). $v = |\mu|$ is possible.

Usually, as in example 2, $R^1_\sigma \neq L^1$; *if however* Ω *is a* δ*-ring, then*

$$L^1(v, B) = R^1_\sigma(v, B) \tag{29}$$

[11], p. 74.

Example 11. X set $\neq \phi$, $\Omega = \{\phi\} \cup \{\{x\} : x \in X\}$, $\mu(\{x\}) := 1$, $S = $ B-space or F as in example 5, $\omega = \mu(A)\, a$, $P = R^+$; then $R^1_0(\mu, S) = T(\Omega, S)$ and

$$R^1_\sigma(\mu, S) = R^1(\mu, S) = L^1(\mu, S) = l^1(X, S).$$

Thus under (15) a theory of abstract L^1-spaces follows from

$$(30) \qquad \mathbf{L}^1(\omega, \mathbf{P}, \mathbf{W}) := R^1(\overline{\omega}, P, W) \quad , \quad \overline{\omega}: Y \times \delta\Omega \to Y'$$

σ-additive, $\supset \omega$, $\Omega \subset \delta\Omega$ δ-ring. L^1 is d_ω-complete and the convergence theorems hold as soon as $F \to f$ ω-a.e. and F d_ω-Cauchy imply $F \Rightarrow f(\omega)$, and $G \to g(\omega)$ implies $G \to g$ ω-a.e. locally for a subnet.

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