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Fisica matematica. — *A note on a variational formulation of the Einstein equations for thermo-elastic materials.* Nota di FRANCO CARDIN (*), presentata (**) dal Socio G. GRIOLI.

RIASSUNTO. — Sulla base di una nota versione relativistica della disuguaglianza di Clausius-Duhem si deduce in Relatività Generale che, anche se per un corpo termo-elastico $\mathscr{C}$ la legge di Fourier non è lineare, il tensore di Fourier è definito positivo. Lo scopo principale del lavoro consiste nello stabilire l'equivalenza delle equazioni gravitazionali di Einstein per $\mathscr{C}$ (il cui tensore energia-impulso include il tensore termodinamico di Eckart) con una condizione variazionale, scritta in due versioni equivalenti.

1. INTRODUCTION

In late years a relativistic variational principle due to Schöpf (see [6]), which consists of the equivalence of a certain variational condition with the Einstein gravitational equations for elastic materials, has been extended by Bressan to polar materials in [2] and afterwards by Pitteri to materials of any order $n \geq 2$ in [4] and [5].

In spite of the constitutive complexity of those materials the above principles deal with bodies necessarily undergoing adiabatic processes, i.e. with $D\eta/Ds \equiv 0 \equiv q^{\alpha}$, where η is the specific entropy and q^{α} is the heat flux.

In this note a variational formulation of the Einstein-Eckart equations (so that the energy-momentum tensor $\mathscr{U}_{\alpha\beta}$ includes Eckart's thermodynamic tensor $2\, u_{(\alpha}\, q_{\beta)}$) is stated for thermo-elastic materials in two versions.

More in detail, a brief introduction to the Lagrangian formalism in general relativity (based on [3]) and to the corresponding basic laws is presented in NN 2-3, and there the class TE of thermo-elastic materials is introduced in a slightly modified way with respect to [3], on the basis of a relativistic version of the Clausius-Duhem inequality. Thus a result concerning the linear case and well known in classical physics can be extended, in relativity theory, to the non-linear case. In N° 4 a scalar constraint between $\delta\eta$ and $\delta g_{\alpha\beta}$ (where $g_{\alpha\beta}$ is the metric tensor) is reached by means of simple considerations on the purely adiabatic variational case, and it is called *weakly-isentropic* (variational)

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constraint (cf. (4.6)). It gives rise to the (weakly-isentropic) version of the afore-mentioned variational principle. The other version includes no constraints on $\delta\eta$ and $\delta g_{\alpha\beta}$.

2. Lagrangian formalism for material processes in general relativity

We assume the universe to consist of a continuous body $\mathscr{C}$ moving in a space-time $\mathscr{S}_4$ of general relativity. We regard $\mathscr{C}$ as a set of material particles. Let $\mathscr{P}^*$ be any process physically possible for $\mathscr{C}$, which describes the world tube $\mathscr{W}_{\mathscr{C}}(\mathscr{P}^*)$ in the determination $\overset{*}{\mathscr{S}_4}$ of $\mathscr{S}_4$. Let y^{Σ} (1) $(t = y^0)$ be the co-ordinates of a typical event point $\mathscr{E}$ of $\overset{*}{\mathscr{S}_4}$ in an admissible reference frame (y) (cf. [3], § 15). In this frame we denote the metric tensor for $\mathscr{P}^*$ by $\overset{*}{g}_{\Gamma\Delta} = \overset{*}{\tilde{g}}_{\Gamma\Delta}(y^{\Sigma})$ and represent the world-line $\mathscr{W}_{P^*}(\mathscr{P}^*)$ of $P^* \in \mathscr{C}$ by $y^{\Sigma} = \tilde{y}(P^*, s^*)$. Thus the space-time metric is $(ds^*)^2 = -\overset{*}{g}_{\Gamma\Delta}\, dy^{\Gamma}\, dy^{\Delta}$ and the 4-velocity of P^* at $\bar{y}^{\Sigma} = \tilde{y}^{\Sigma}(P^*, \bar{s}^*)$ is $u^{*\Sigma} = (\partial\tilde{y}^{\Sigma}/\partial s^*)(P^*, \bar{s}^*)$. Let $\overset{\perp *}{g}_{\Gamma\Delta} = \overset{*}{g}_{\Gamma\Delta} + \overset{*}{u}_{\Gamma}\overset{*}{u}_{\Delta}$ be the spatial projector. Following [3] the L-*th material* (or *Lagrangian*) co-ordinate of $P^* \in \mathscr{C}$ is the co-ordinate y^L of the intersection of the world-line $\mathscr{W}_{P^*}(\mathscr{P}^*)$ of P^* with the hyperplane $t = 0$, say σ_3. Let $\overset{*}{\mathscr{S}_3} = \mathscr{W}_{\mathscr{C}}(\mathscr{P}^*) \cap \sigma_3$. For a sufficiently regular motion for $\mathscr{C}$ there is an one-to-one mapping from $\mathscr{C}$ to $\overset{*}{\mathscr{S}_3}$, $y^L = \tilde{y}^L(P^*)$. We now call the physical state Σ^* (cf. [3], p. 139) and the configuration of $\mathscr{C}$ in $\overset{*}{\mathscr{S}_3}$ within the process $\mathscr{P}^*$ *reference physical state* and *reference configuration* respectively. We denote the spatial metric on $\overset{*}{\mathscr{S}_3}$ by

$$(2.1) \qquad (d\overset{\perp *}{s})^2 = \overset{*}{a}_{LM}\, dy^L\, dy^M\,, \qquad \text{where}$$

$$\overset{*}{a}_{LM} = \overset{*}{\tilde{a}}_{LM}(y^1, y^2, y^3) = \overset{\perp *}{g}_{LM}(0\,, y^1, y^2, y^3)\,,$$

and call $(d\overset{\perp *}{s})^2$, which depends only on $\mathscr{P}^*$ and $\overset{*}{\mathscr{S}_3}$, *material metric*.

* * *

Let now x^{ρ} be the co-ordinates of a typical event point $\mathscr{E}$ of $\mathscr{S}_4$ in an admissible reference frame (x), and let $\mathscr{E}$ belong to the world-tube $\mathscr{W}_{\mathscr{C}}(\mathscr{P})$ of a physically possible process $\mathscr{P}$ for $\mathscr{C}$. This is characterized by (*i*) $\mathscr{C}$'s *motion* $\mathscr{M}$:

$$(2.2) \qquad x^{\alpha} = \tilde{x}^{\alpha}(t\,, y^L)\,,$$

(*ii*) the *metric tensor* $\tilde{g}_{\alpha\beta}(x^{\rho})$, and (*iii*) *some fields* $\tilde{m}_1(x^{\alpha})\,,\cdots,\tilde{m}_N(x^{\alpha})$ such that, for $t \in \mathbf{R}$ and $y^L \in \overset{*}{\mathscr{S}_3}$, the values $\tilde{m}_i[\tilde{x}^{\alpha}(t\,,y^L)]\,, i = 1\,,\cdots, N$, equal the physical values of some magnitudes $\mathscr{M}_1\,,\cdots,\mathscr{M}_N$, relevant for the physical state $d\Sigma$ at the event point $\tilde{x}^{\alpha}(t\,,y^L)$ of the matter element $d\mathscr{C}$ that contains P^*.

(1) Greek [Latin] letters run from 0 [1] to 3.

In connection with the process $\mathscr{P}$ we recall that $ds^2 = -g_{\alpha\beta}\, dx^\alpha\, ds^\beta$ has the signature $+2$, $u^\alpha = D\tilde{x}^\alpha/Ds$ (2) $(u^\alpha u_\alpha = -1)$ $[A^\alpha = Du^\alpha/Ds\ (u_\alpha A^\alpha = 0)]$ is the 4-velocity [4-acceleration], $\overset{\perp}{g}_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ is the spatial projector, and $x^\rho{}_L = \partial\tilde{x}^\rho/\partial y^L$ is the *position gradient*. The (*first-right*) *Cauchy-Green* tensor C_{LM} and the *strain* tensor ε_{LM} are defined by

$$\overset{*}{a}_{LM} + 2\,\varepsilon_{LM} = C_{LM} = \alpha_{\rho L}\,\alpha^\rho{}_M\,, \qquad \text{where} \quad \alpha^\rho{}_L = \overset{\perp}{g}{}^\rho{}_\sigma\, x^\sigma{}_L\,, \tag{2.3}$$

and

$$\frac{DC_{LM}}{Ds} = 2\,u_{(\rho/\sigma)}\,\alpha^\rho_L\,\alpha^\sigma_M = 2\,\frac{D\varepsilon_{LM}}{Ds} \tag{2.4}$$

hold (cf. [3], $(57.6)_3$).

Now we consider the matter element $d\mathscr{C}$ containing the material point $y^L (= \breve{y}^L(P^*))$, its *proper volume* $dC^* (> 0)$ in the reference configuration $\mathscr{S}^*_3$, and its actual counterpart dC (at $\boldsymbol{x} \in \mathscr{W}_{P^*}(\mathscr{P})$). Then (cf. [3], § 56)

$$\mathscr{D} = \frac{dC}{dC^*}\,, \qquad \text{where} \quad \mathscr{D} = \sqrt{\frac{-g}{a^*}}\,\left|\frac{\partial(x^0,\cdots,x^3)}{\partial(t,y^1,y^2,y^3)}\right|\frac{Dt}{Ds} > 0 \tag{2.5}$$

$$(g = \det\|g_{\alpha\beta}\|\,,\ a^* = \det\|\overset{*}{a}_{LM}\|)\,.$$

Let $c^2\, dm^* = c^2 k^*\, dC^*$ (3) be the *proper gravitational mass* of $d\mathscr{C}$ (in energy units) in $\mathscr{S}^*_3$, so that k^* is the proper density in the reference configuration; let $k(\boldsymbol{x})$ be the (actual) *proper density of conventional mass* (in $\mathscr{P}$):

$$k\, dC = k^*\, dC^*\,; \tag{2.6}$$

hence the Lagrangian and Eulerian *equations of continuity*

$$k^* = \mathscr{D}k \quad , \quad (ku^\alpha)_{/\alpha} = 0 \tag{2.7}$$

hold for it; finally, let $c^2\, dm = \rho\, dC$ be the (actual) *proper gravitational mass* of $d\mathscr{C}$ at $\boldsymbol{x} \in \mathscr{W}_{P^*}(\mathscr{P})$, where $\rho(\boldsymbol{x})$ is the proper density of gravitational mass. After [3] (cf. § 21), we define

$$kw\, dC = c^2\, dm - c^2\, dm^* = (\rho - kc^2)\, dC\,, \tag{2.8}$$

where w is briefly called the *specific internal energy* (of $\mathscr{C}$ at P^*).

(2) By $T^{\cdots}_{\cdots,\alpha}$ $[T^{\cdots}_{\cdots/\alpha}]$ we mean the partial [covariant] derivative of $T^{\cdots}_{\cdots}$, and we use the following notations:

$$2\,T^{[\alpha\beta]} = T^{\alpha\beta} - T^{\beta\alpha} \quad , \quad 2\,T^{(\alpha\beta)} = T^{\alpha\beta} + T^{\beta\alpha} \quad , \quad DT^{\cdots}_{\cdots}/Ds = T^{\cdots}_{\cdots/\alpha}\, u^\alpha.$$

(3) c is the velocity of light in vacuo.

3. Basic laws. Thermo–elastic materials

We consider a body $\mathscr{C}$ uncapable of electromagnetic phenomena, having the energy–momentum tensor:

$$\mathscr{U}_{\alpha\beta} = \rho u_\alpha u_\beta + X_{\alpha\beta} + Q_{\alpha\beta} , \qquad \text{with} \quad Q_{\alpha\beta} = 2\, u_{(\alpha} q_{\beta)} , \tag{3.1}$$

where $X_{\alpha\beta}$ is the *Cauchy stress* tensor ($X_{[\alpha\beta]} = 0 = X_{\alpha\beta} u^\beta$) and q_α is the (spatial) *heat flux* vector. We postulate the *Einstein–Eckart gravitational equations* in the form:

$$A_{\alpha\beta} + \frac{8\,\pi h}{c^4} \mathscr{U}_{\alpha\beta} = 0 \qquad (A_{\alpha\beta} = R_{\alpha\rho}{}^{\rho}{}_{\beta} + \tfrac{1}{2} R_{\rho\sigma}{}^{\rho\sigma} g_{\alpha\beta}) , \tag{3.2}$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann's tensor and h is the Cavendish's constant. The conservation equations $\mathscr{U}^{\alpha\beta}{}_{/\beta} = 0$, which follow from (3.2) by $A^{\alpha\beta}{}_{/\beta} = 0$, include the *energy conservation equation* $u_\alpha \mathscr{U}^{\alpha\beta}{}_{/\beta} = 0$, which reads

$$k \frac{Dw}{Ds} + X^{\alpha\beta} u_{(\alpha/\beta)} = - q^\alpha{}_{/\alpha} - q^\alpha A_\alpha . \tag{3.3}$$

By regarding $X^{\alpha\beta} u_{(\alpha/\beta)}$ as the *work power of internal forces*, i.e. $\delta l^{(i)}/Ds = X^{\alpha\beta} u_{(\alpha/\beta)}$, on the basis of the 1st *principle of thermodynamics* we interprete the quantity $- q^\alpha{}_{/\alpha} - q^\alpha A_\alpha$ as the *heat absorbed* by $\mathscr{C}$ (per unit of proper volume and time), hence (cf. [3], § 24),

$$k \frac{Dw}{Ds} + \frac{\delta l^{(i)}}{Ds} = Q_{\text{ass}} = - q^\alpha{}_{/\alpha} - q^\alpha A_\alpha . \tag{3.4}$$

Let Y^{LM} be the *second Piola–Kirchhoff stress tensor*; it is related to $X^{\rho\sigma}$ by

$$X^{\rho\sigma} = \frac{1}{\mathscr{D}} \alpha^\rho{}_L\, \alpha^\sigma{}_M\, Y^{LM} . \tag{3.5}$$

Let $T\,(> 0)$ $[\eta]$ be the *absolute temperature* [*specific entropy*]. We assume the absence of the electromagnetic field, and, as the 2nd *principle of thermodynamics*, we introduce the following relativistic version of the *Clausius–Duhem inequality*:

Under the definitions

$$\mathscr{S}^\alpha = k\eta u^\alpha + s^\alpha \quad , \quad s^\alpha = \frac{q^\alpha}{T} (= \overset{1}{\mathscr{S}}{}^\alpha) , \tag{3.6}$$

the inequality

$$\mathscr{S}^\alpha{}_{/\alpha} \geq 0 \tag{3.7}$$

must be satisfied by $\mathscr{C}$ *along every physically possible process* (4).

(4) The relativization (3.7) of the Clausius–Duhem inequality is supported in [3], App. C, by means of kinematical considerations, see also [1].

By $(3.6)_{1,2}$, (3.3), (3.5), $(2.4)_2$, $(2.7)_1$, and $(2.5)_3$, (3.7) can be written in a Lagrangian form:

$$(3.8) \qquad k^* \frac{D\eta}{Ds} - \frac{k^*}{T}\frac{Dw}{Ds} - \frac{Y^{LM}}{T}\frac{D\varepsilon_{LM}}{Ds} - \frac{\mathscr{D}q^\alpha}{T^2}(T_{/\alpha} + TA_\alpha) \geq 0 .$$

We say that the body is *thermo-elastic* (TE) at its materials point P* (or y^L) if the following two conditions hold at P* (5):

(A) *The specific internal energy* w, *the specific entropy* η, *and* Y^{LM}, *are twice continuously differentiable functions of* ε_{LM} *and* T*; the heat flux* q^α *is a twice continuously differentiable vector valued function of* ε_{LM}, T, $T_{/\underset{\alpha}{1}}$, *and* A_α (6).

(B) *Constraints are absent, in the (narrow) sense that admissible values of* ε_{LM} *and* T *are physically compatible with arbitrary values of* $T_{/\alpha}$, $D\varepsilon_{LM}/Ds$, *and* A_α.

Let $\mathscr{F} = w - T\eta$ be the *specific free energy*. Then by (A) $\mathscr{F} = \breve{\mathscr{F}}(y^L, \varepsilon_{LM}, T)$ and (3.8) becomes

$$(3.9) \qquad k^*\left(\frac{\partial\breve{\mathscr{F}}}{\partial T} + \eta\right)\frac{DT}{Ds} + \left(k^*\frac{\partial\breve{\mathscr{F}}}{\partial\varepsilon_{LM}} + Y^{LM}\right)\frac{D\varepsilon_{LM}}{Ds} + \frac{\mathscr{D}q^\alpha}{T}(T_{/\alpha} + TA_\alpha) \leq 0.$$

By condition (B) the values of DT/Ds, $T_{/\underset{\alpha}{1}}$, $D\varepsilon_{LM}/Ds$, and A_α are arbitrary, so that (3.9) yields

$$(3.10) \qquad \eta = -\frac{\partial\breve{\mathscr{F}}}{\partial T}(y^L, \varepsilon_{LM}, T) \quad , \quad Y^{LM} = -k^*\frac{\partial\breve{\mathscr{F}}}{\partial\varepsilon_{LM}}(y^L, \varepsilon_{LM}, T),$$

and (3.9) simplifies into

$$(3.11) \qquad q^\alpha(T_{/\alpha} + TA_\alpha) \leq 0 .$$

By simple and well known considerations based on Helmholtz postulate (cf. [3], § 28) $\partial^2\breve{\mathscr{F}}/\partial T^2 < 0$ holds; hence $(3.10)_1$ defines T as an implicit function $\tilde{T}$ of y^L, η, and ε_{LM}. Set $\tilde{w}(y^L, \varepsilon_{LM}, \eta) = \breve{w}[y^L, \varepsilon_{LM}, \tilde{T}(y^L, \varepsilon_{LM}, \eta)]$. By a deduction like the above one we derive

$$(3.12) \qquad T = \frac{\partial\tilde{w}}{\partial\eta}(y^L, \varepsilon_{LM}, \eta) \quad , \quad Y^{LM} = -k^*\frac{\partial\tilde{w}}{\partial\varepsilon_{LM}}(y^L, \varepsilon_{LM}, \eta) .$$

(5) This definition is a slightly modified version of Bressan's definition of elastic materials in [3], § 63.

(6) A possible dependence of q^α on the 4-acceleration A_α is suggested by the expression $(3.4)_2$ for the absorbed heat Q_{ass} by P*.

* * *

Now suppose that the following expression (possibly *non-linear* in $T_{/\underset{\alpha}{\perp}}$) holds for q^{α}:

$$(3.13) \qquad q^{\alpha} = -[\tilde{\varkappa}^{\alpha\beta}(y^{L}, \varepsilon_{LM}, T, T_{/\underset{\rho}{\perp}})](T_{/\underset{\beta}{\perp}} + TA_{\beta}), \qquad \varkappa^{\alpha\beta} u_{\beta} = 0 = u_{\alpha}\varkappa^{\alpha\beta};$$

then, for fixed values of ε_{LM}, T, and $T_{/\underset{\rho}{\perp}}$ (see (B)), and for arbitrary values of A_{α}, hence of $T_{/\underset{\alpha}{\perp}} + TA_{\alpha}$, the validity of inequality (3.11) implies that $\tilde{\varkappa}^{\alpha\beta}$ is *positive definite*.

Note that in the classical case the analogous result is deduced only for $\partial\tilde{\varkappa}^{ij}/\partial T_{,k} = 0$.

* * *

Let $\mathscr{C} \in TE$ and let (x) be an admissible reference frame. Then we call thermodynamic process $\mathscr{P}$ for $\mathscr{C}$ any triple

$$(3.14) \qquad \mathscr{P} = \langle \boldsymbol{g}, \mathscr{M}, \eta \rangle$$

where $\boldsymbol{g}$, $\mathscr{M}$, and η are the fields $\tilde{g}_{\alpha\beta}(x^{\rho})$, $\tilde{x}^{\alpha}(t, y^{L})$, and $\tilde{\eta}(x^{\rho})$ that solve the Einstein-Eckart equations (3.1,2).

4. A variational principle

It is known (cf. [3], § 69) that the Einstein equations (3.1,2) for $\mathscr{C}$ ($\in$ TE) *uncapable of heat conduction* ($Q^{\alpha\beta} \equiv 0$) are equivalent to the *variational condition*:

$$(4.1) \qquad \delta_{\mathscr{P}} F[\mathscr{P}] = 0 \qquad \textit{for all} \quad \delta\mathscr{P} = \langle \delta\boldsymbol{g}, 0, 0 \rangle,$$

where

$$(4.2) \qquad F[\mathscr{P}] = \int_{\Omega} \sqrt{-g}\left(R + \frac{16\pi h}{c^4}\rho\right) dx \quad (R = R_{\alpha\beta}{}^{\alpha\beta},\ dx = dx^0 \cdots dx^3),$$

$\delta g_{\alpha\beta} \in C^{(2)}_{(\Omega)}$ *and* $\delta g_{\alpha\beta} = 0 = \delta g_{\alpha\beta,\gamma}$ *on the frontier* $\mathscr{F}\Omega$ *of the domain* $\Omega \subset \mathscr{W}_{\mathscr{C}}(\mathscr{P})$.

The variations $\delta\mathscr{P}$ considered above are *isentropic* ($\delta_{\mathscr{P}}\eta = 0$); it is justified by the fact that, for $q^{\alpha} \equiv 0$, the energy conservation equation (3.3) becomes $D\eta/Ds = 0$ by (3.12), (3.5), and (2.4); hence $\eta = \tilde{\eta}(y^{L})$.

Now in connection with a general body $\mathscr{C} \in TE$ we consider the vector:

$$(4.3) \qquad \mathscr{S}^{*\alpha} = \mathscr{D}\mathscr{S}^{\alpha} = k^{*}\eta u^{\alpha} + s^{*\alpha}, \quad s^{*\alpha} = \mathscr{D}s^{\alpha} \qquad \text{(see (3.6))},$$

and its variation for some $\delta\mathscr{P} = \langle \delta \boldsymbol{g}\, , \mathrm{o}\, , \delta\eta \rangle$:

$$(4.4) \qquad \delta_{\mathscr{P}}\, \mathscr{S}^{*\alpha} = k^{*}\, \eta\delta_{\mathscr{P}}\, u^{\alpha} + k^{\alpha}\, u^{\alpha}\, \delta\eta + \delta_{\mathscr{P}}\, s^{*\alpha}\, .$$

In particular we have:

$$(4.5) \qquad u_{\alpha}\, \delta_{\mathscr{P}}\, \mathscr{S}^{*\alpha} = k^{*}\, \eta u_{\alpha}\, \delta_{\mathscr{P}}\, u^{\alpha} - k^{*}\, \delta\eta + u_{\alpha}\, \delta_{\mathscr{P}}\, s^{*\alpha}\, .$$

If $q^{\alpha} \equiv \mathrm{o}$ holds for $\mathscr{C}$ then by $(4.3)_3$ and $(3.6)_2$ we have $\delta_{\mathscr{P}}\, s^{*\alpha} = \mathrm{o}$ and, in this case, by (4.5) we note that the relation

$$(4.6) \qquad u_{\alpha}\, \delta_{\mathscr{P}}\, \mathscr{S}^{*\alpha} = k^{*}\, \eta u_{\alpha}\, \delta_{\mathscr{P}}\, u^{\alpha}$$

characterizes the isentropic variations $\delta\mathscr{P}$ (for which $\delta\eta \equiv \mathrm{o}$). Since this characterization holds only in the case $q^{\alpha} \equiv \mathrm{o}$, (4.6) will be called the *weakly isentropic variational condition*. We will use it in a basic way.

* * *

Now we say that the process $\mathscr{P} = \langle \boldsymbol{g}\, , \mathscr{M}\, , \eta \rangle$ for $\mathscr{C} \in \mathrm{TE}$ satisfies the *weakly isentropic variational condition* $\mathrm{VC}_{\mathscr{P}}^{(\mathrm{w.i.})}$ [*unrestricted variational condition* $\mathrm{VC}_{\mathscr{P}}^{(\mathrm{un.})}$] *in* Ω, *if for any variation* $\delta\mathscr{P} = \langle \delta \boldsymbol{g}\, , \mathrm{o}\, , \delta\eta \rangle$ *of* $\mathscr{P}$ *that fulfils the conditions*

$$(4.7) \qquad \delta g_{\alpha\beta} \in \mathrm{C}^{(2)}(\Omega) \quad , \quad \delta g_{\alpha\beta} = \mathrm{o} = \delta g_{\alpha\beta,\gamma} \quad \mathit{on} \quad \mathscr{F}\Omega\, ,$$

and (4.6) [*and for arbitrary variation* $\delta\eta \in \mathrm{C}^{(0)}(\Omega)$] *in* Ω, *we have the first* [*second*] *of the relations* (see (4.2)):

$$(4.8) \qquad \delta_{\mathscr{P}}\, \mathrm{F}\, [\mathscr{P}] = \mathrm{o}\, ,$$

$$(4.8') \qquad \delta_{\mathscr{P}}\, \mathrm{F}\, [\mathscr{P}] = \frac{16\, \pi h}{c^{4}} \int_{\Omega} \frac{\mathrm{T}}{\mathscr{D}}\, u_{\alpha}\, (k^{*}\, \eta\delta_{\mathscr{P}}\, u^{\alpha} - \delta_{\mathscr{P}}\, \mathscr{S}^{*\alpha})\, \sqrt{-g}\; dx\, .$$

THEOREM. *A continuous body* $\mathscr{C} \in \mathrm{TE}$ *satisfies the variational condition* $\mathrm{VC}_{\mathscr{P}}^{(\mathrm{w.i.})}$ [$\mathrm{VC}_{\mathscr{P}}^{(\mathrm{un.})}$] *in* Ω *iff the Einstein-Eckart equations* (3.1,2) *hold for it in* Ω.

Proof. The following formulas for arbitrary $\delta g_{\alpha\beta}$ and $\delta\eta$, well known in the adiabatic case (cf. [3], § 68), are still holding because the varied quantities do not depend on η:

$$(4.9) \quad \begin{cases} \delta_{\mathscr{P}}\, u^{\alpha} = \frac{1}{2}\, u^{\alpha}\, u^{\rho}\, u^{\sigma}\, \delta g_{\rho\sigma} \quad , \quad \delta_{\mathscr{P}}\, u_{\alpha} = u^{\sigma}\, \delta g_{(\alpha\sigma)} + \frac{1}{2}\, u_{\alpha}\, u^{\rho}\, u^{\sigma}\, \delta g_{\rho\sigma}\, , \\ \delta_{\mathscr{P}}\, \sqrt{-g} = \frac{1}{2}\sqrt{-g}\; g^{\rho\sigma}\, \delta g_{\rho\sigma} \quad , \quad \delta_{\mathscr{P}}\, k = -\frac{1}{2}\, k \overset{\perp}{g}{}^{\rho\sigma}\, \delta g_{\rho\sigma}\, , \\ \qquad \delta_{\mathscr{P}}\, \varepsilon_{\mathrm{LM}} = \frac{1}{2}\, \alpha_{\mathrm{L}}^{\rho}\, \alpha_{\mathrm{M}}^{\sigma}\, \delta g_{\rho\sigma}\, . \end{cases}$$

Since $s^{*\alpha} u_\alpha = 0$, whence $\delta_{\mathscr{P}}(s^{*\alpha} u_\alpha) = 0$, we have $u_\alpha \delta_{\mathscr{P}} s^{*\alpha} = -s^{*\alpha} \delta_{\mathscr{P}} u_\alpha$, and, by using $(4.3)_3$ and $(4.9)_2$, relation (4.5) becomes

$$k^* \delta\eta = k^* \eta u_\alpha \delta_{\mathscr{P}} u^\alpha - u_\alpha \delta_{\mathscr{P}} \mathscr{S}^{*\alpha} - \mathscr{D} s^{(\alpha} u^{\beta)} \delta g_{\alpha\beta} . \tag{4.10}$$

For $\delta g_{\alpha\beta}$, fulfilling (4.7) we have, as is well known (cf. [3], (69.12)),

$$\delta_{\mathscr{P}} \int_\Omega \sqrt{-g}\, R\, dx = - \int_\Omega A^{\alpha\beta} \delta g_{\alpha\beta} \sqrt{-g}\, dx . \tag{4.11}$$

Moreover

$$\delta_{\mathscr{P}}(\sqrt{-g}\,\rho) = (w + c^2)\, \delta_{\mathscr{P}}(\sqrt{-g}\, k) + \sqrt{-g}\, k \delta_{\mathscr{P}} w . \tag{4.12}$$

and by $(4.9)_{3,4}$ we have

$$(w + c^2)\, \delta_{\mathscr{P}}(\sqrt{-g}\, k) = - \frac{\sqrt{-g}}{2} \rho u^\alpha u^\beta \delta g_{\alpha\beta} , \tag{4.13}$$

while, by (3.12), $(4.9)_5$, (3.5), and (4.10),

$$\begin{aligned}
\sqrt{-g}\, k \delta_{\mathscr{P}} w &= \sqrt{-g}\, k \left(\frac{\partial \tilde{w}}{\partial \varepsilon_{LM}} \delta_{\mathscr{P}} \varepsilon_{LM} + \frac{\partial \tilde{w}}{\partial \eta} \delta\eta \right) \\
&= \sqrt{-g}\, k \left(-\frac{1}{2} \frac{Y^{LM}}{k^*} \alpha^\alpha_L \alpha^\beta_M \delta g_{\alpha\beta} + T\, \delta\eta \right) \\
&= \sqrt{-g}\, k \Big[- \frac{1}{2k} X^{\alpha\beta} \delta g_{\alpha\beta} + \frac{T}{k^*} (- \mathscr{D} s^{(\alpha} u^{\beta)} \delta g_{\alpha\beta} + \\
&\qquad + k^* \eta u_\alpha \delta_{\mathscr{P}} u^\alpha - u_\alpha \delta_{\mathscr{P}} \mathscr{S}^{*\alpha}) \Big] ,
\end{aligned}$$

$$\begin{aligned}
\sqrt{-g}\, k \delta_{\mathscr{P}} w = &- \frac{\sqrt{-g}}{2} (X^{\alpha\beta} + 2\, T s^{(\alpha} u^{\beta)}) \delta g_{\alpha\beta} + \\
&+ \sqrt{-g}\, \frac{T}{\mathscr{D}} (k^* \eta u_\alpha \delta_{\mathscr{P}} u^\alpha - u_\alpha \delta_{\mathscr{P}} \mathscr{S}^{*\alpha}) .
\end{aligned} \tag{4.14}$$

Remembering that $s^\alpha = q^\alpha T^{-1}$ and $Q^{\alpha\beta} = 2\, q^{(\alpha} u^{\beta)}$, by (4.11–14) we have:

$$\begin{aligned}
\delta_{\mathscr{P}} F[\mathscr{P}] = &- \int_\Omega \sqrt{-g} \left[A^{\alpha\beta} + \frac{8\pi h}{c^4} (\rho u^\alpha u^\beta + X^{\alpha\beta} + Q^{\alpha\beta}) \right] \delta g_{\alpha\beta}\, dx + \\
&+ \frac{16\pi h}{c^4} \int_\Omega \sqrt{-g}\, \frac{T}{\mathscr{D}} u_\alpha (k^* \eta \delta_{\mathscr{P}} u^\alpha - \delta_{\mathscr{P}} \mathscr{S}^{*\alpha})\, dx .
\end{aligned} \tag{4.15}$$

By the arbitrariness of $\delta g_{\alpha\beta}$ and the continuity of $\left[A^{\alpha\beta} + \frac{8\pi h}{c^4}(\rho u^\alpha u^\beta + \cdots)\right]$, the two versions of the theorem easily follow.

q. e. d.

The theorem above can be extended in a natural way to more general classes of materials, such as the polar materials [2] and the materials of order $n \geq 2$ [5].

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