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Complementary distributions and Pontrjagin classes

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Geometria differenziale. — *Complementary distributions and Pontrjagin classes* (*). Nota (**) di IDA CATTANEO GASPARINI e GIUSEPPE DE CECCO, presentata dal Socio straniero A. LICHNEROWICZ.

RIASSUNTO. — Da una condizione necessaria per l'esistenza di k (> 2) distribuzioni complementari su una varietà si deducono legami tra le classi di Pontrjagin delle distribuzioni e dei fibrati trasversi.

Let M be a riemannian smooth orientable manifold of dimension n (even or odd) and suppose that M has k complementary (smooth) distributions of oriented n_i-planes ($i = 1, \cdots, k$); i.e. for every point $p \in M$ the tangent space $T_p(M)$ can be decomposed into the direct sum of the subspaces $T_p^1, \cdots, T_p^k$ of $T_p(M)$ where $\dim T_p^i = n_i$ (and hence $n_1 + \cdots + n_k = n$).

Then one says that M admits an " almost product " or " almost multiproduct " structure.

In a paper of 1969 [3] one of the present authors showed that the vanishing of certain Pontrjagin classes is a necessary condition for the existence of k complementary distributions on M. After a review of these results we prove the following

THEOREM A. *Let* M *be a riemannian smooth orientable manifold of dimension* n. *Furthermore let* M *have* k *complementary distributions* $E_i \subset TM$ *and let the bundle* $Q_i = TM/E_i$ *have fibre of dimension* q_i *with*

$$q_i = n - n_i \quad \text{and} \quad n_1 + \cdots + n_k = n .$$

Finally let $p_r(E) \in H^{4r}(M ; \mathbf{R})$ *denote the r-th real Pontrjagin class of the bundle* E. *Then if*

$$p_h(Q_i)\, p_s(E_i) = 0 \qquad \forall h, s \geq 1 \quad \text{and} \quad h + s = r$$

one has

$$p_r(Q_i) = 0 \qquad 2r > \max(n_1, \cdots, n_k) .$$

Using Bott's " Vanishing Theorem " one can, in a certain sense invert the preceding result obtaining

THEOREM B. *Let* M *be a riemannian smooth orientable manifold of dimension* n. *Furthermore let* M *have* k *complementary distributions* $E_i \subset TM$

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and let the bundle $Q_i = TM/E_i$ *have fibre of dimension* q_i *with*

$$q_i = n - n_i \qquad \text{and} \quad n_1 + \cdots + n_k = n .$$

If Q_i *is integrable, then for* $2r > \max(n_1, \cdots, n_k, 4q_i)$

$$p_h(Q_i)\, p_s(E_i) = 0 \qquad \forall h, s \geq 1 \quad \text{and } h + s = r$$

holds.

In order to give a self-contained presentation we recall the necessary preliminaries.

1. PRELIMINARIES

Let M be a smooth (paracompact) manifold and let E be a vector $\mathbf{R}^q$-bundle on M. As is well known, the total (real) Pontrjagin class $p(E)$ of E is defined by

$$p(E) = 1 + p_1(E) + \cdots + p_{[q/2]}(E) = \left[\det\left(I - \frac{1}{2\pi}\Omega\right)\right]$$

where Ω is the curvature of an arbitrary connection on E and

$$p_r(E) \in H^{4r}(M; \mathbf{R}),$$

where $H^*(M; \mathbf{R})$ is the de Rham cohomology ring of M. $p_r(E)$ is called the *r-the Pontrjagin class of* E. Clearly $p_r(E) = 0$ for $4r > n$ if $n = \dim M$. Moreover if E is an oriented bundle of even dimension q, then the class $p_{q/2}(E)$, which is locally represented by the closed form $(2\pi)^{-q} \det \Omega$, equals the square of the Euler class; this latter is strictly connected with the Euler-Poincaré characteristic of the manifold under consideration, if $E = TM$.

If E is the tangent bundle TM, then the classes are also called *Pontrjagin classes of* M and are often denoted by $P_r(M)$.

Let Ω be the curvature for of a connection on the principal fibre bundle of orthonormal frames, then the explicit expression of Pontrjagin classes is given by (1)

$$p_r(TM) = \left[\frac{[(2r)!]^2}{(2^r r!)^2 (2\pi)^{2r}} \sum_{(i)} \Theta^{(2r)}_{i_1 \cdots i_{2r}} \wedge \Theta^{(2r)}_{i_1 \cdots i_{2r}}\right] \tag{1.1}$$

where

$$\Theta^{(s)}_{i_1 \cdots i_s} = \frac{1}{s!} \sum_{(j)} \delta(i_1, \cdots, i_s; j_1, \cdots, j_s)\, \Omega_{j_1 j_2} \wedge \cdots \wedge \Omega_{j_{s-1} j_s} \tag{1.2}$$

s is an even integer and $\delta(i_1 \cdots, i_s; j_1, \cdots, j_s)$ is the generalised Kronecker symbol.

(1) See J. A. Thorpe [6].

For n even, the n-form

$$\Omega^{(n)} = \left[2^n \pi^{n/2} \left(\frac{n}{2}\right)!\right]^{-1} \Sigma\varepsilon_{i_1 \cdots i_n} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{n-1} i_n} \tag{1.3}$$

called *Gauss curvature form* of M, is a representative of the Euler class of TM and if M is compact and orientable, the Gauss-Bonnet theorem says that

$$\int_M f^* \Omega^{(n)} = \chi(M)$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M and $f: M \to \tilde{G}_n(M)$ is an orientation of M , $\tilde{G}_n(M)$ being the Grassmann bundle of the oriented n-planes tangent to M.

2. A VANISHING THEOREM

We can now prove the following theorem [3].

THEOREM (2.1). *Let* M *be a riemannian smooth orientable manifold of dimension* n *which admits* k *complementary (smooth) distributions of oriented* n_i*-planes* $(i = 1, \cdots, k)$. *Then the real Pontrjagin classes* $P_r(M)$ *are null for* $2r > \max(n_1, \cdots, n_k)$.

Proof. Let $\tilde{E}$ be the principal fibre bundle of the orthonormal frames (associated to the tangent). Its structural group is $\tilde{G} = SO(n)$ (the rotation group). We recall that the Lie algebra $\mathfrak{G}$ of $SO(n)$ can be identified with the space of the skew-symmetric matrices of order n.

Let us consider the subbundle E of $\tilde{E}$ formed of the frames "adapted" to the distributions, viz. the orthogonal frames $\{e_i\}$ $(i = 1, \cdots, n)$ so that the vectors

$$\{e_{\alpha_j}\} \; \alpha_j = n_1 + \cdots + n_{j-1} + 1, \cdots, n_1 + \cdots + n_j \qquad (n_0 = 0)$$

form a basis for T^j. The bundle E can be regarded as having structural group

$$G = SO(n_1) \times SO(n_2) \times \cdots \times SO(n_k).$$

A connection ω on E is represented by a 1-form which takes values in the Lie algebra $\mathfrak{G}$ of G, where $\mathfrak{G}$ is the direct product of the Lie algebras of $SO(n_r)$. Hence one obtains

$$\omega_{ii} = 0 \qquad \omega_{ij} + \omega_{ji} = 0 \qquad i, j = 1, \cdots, n$$

$$\omega_{\alpha_i \beta_j} = 0 \qquad i \neq j \qquad i, j = 1, \cdots, k;$$

$$\alpha_i, \beta_j = n_1 + \cdots + n_{j-1} + 1, \cdots, n_1 + \cdots + n_j.$$

Analogous relations hold for the components of the curvature form Ω. Therefore, if $2r > \max(n_1, \cdots, n_k)$, then each term in (1.2) will have a factor $\Omega_{\alpha_i \beta_j}$ with $i \neq j$; the assertion $p_r(\mathrm{TM}) = \mathrm{P}_r(\mathrm{M}) = 0$ then follows from (1.1).

Remarks (2.2). If $k = n$ and therefore $n_1 = 1 \ \forall i$ (i.e. the manifold is parallelizable) then $\mathrm{P}_r(\mathrm{M}) = 0 \ \forall r$. It follows that the curvature vanishes. The manifold is then flat.

(2.3). For $k = 2$ (and obviously for $k = 1$) the theorem is not meaningful. As for $n_1 + n_2 = n$, $\max(n_1, n_2) \geq [n/2]$ and consequently $\mathrm{P}_r(\mathrm{M}) = 0$ $\forall 2r > \max(n_1, n_2)$.

(2.4). It is worth noticing that the existence of a distribution of q-planes implies the existence of a distribution of $(n - q)$-planes. An argument analogous to the one followed above, using the Gauss curvature form, yields the following

THEOREM (2.5). *Let* M *be a smooth compact orientable manifold of dimension even n and suppose that it has a distribution of oriented q-planes with q odd* ($1 \leq q < n$). *Then the Euler-Poincaré characteristic of* M *is null.*

3. PROOF OF THEOREMS A AND B

(3.1). To every distribution there corresponds a smooth subbundle E_i of the tangent bundle with the fibre of dimension n_i. Denote the quotient bundle by $\mathrm{Q}_i = \mathrm{TM}/\mathrm{E}_i$. From the Whitney duality formula $p(\mathrm{TM}) = p(\mathrm{Q}_i)\, p(\mathrm{E}_i)$ it follows that

$$(3.2) \qquad p_r(\mathrm{TM}) = p_r(\mathrm{Q}_i) + p_{r-1}(\mathrm{Q}_i)\, p_1(\mathrm{E}_i) + \cdots + p_1(\mathrm{Q}_i)\, p_{r-1}(\mathrm{E}_i) + p_r(\mathrm{E}_i)$$

where the product between classes is the " cup product " in the ring $\mathrm{H}^*(\mathrm{M}; \mathbf{R})$.

If $2r > n_i$, then $r > n_i/2$ and hence $p_r(\mathrm{E}_i) = 0$. If moreover

$$(3.3) \qquad p_h(\mathrm{Q}_i)\, p_s(\mathrm{E}_i) = 0 \qquad \forall h, s \geq 1, h + s = r$$

then

$$p_r(\mathrm{TM}) = p_r(\mathrm{Q}_i) \qquad 2r > n_i .$$

Notice that, since the Pontrjagin ring may have divisors of zero, condition (3.3) does not imply that either $p_h(\mathrm{Q}_i) = 0$ or $p_s(\mathrm{E}_i) = 0$. On account of our assumptions, from theorem (1.1) one can conclude that

$$p_r(\mathrm{Q}_i) = 0 \qquad 2r > \max(n_1, \cdots, n_k).$$

This proves theorem A.

(3.4). It is well known that if $E_i \subset TM$ is isomorphic to an integrable subbundle, then by Bott's theorem

$$p_r(Q_i) = 0 \qquad r > 2q_i$$

where $q_i = n - n_i$. Hence in the assumptions of theorem A, if

$$m = \max(n_1, \cdots, n_k) < 4q_i,$$

then one has for the integers h for which $m < h < 4q_i$

$$p_r(Q_i) = 0 \qquad 2r > h$$

without assuming that Q_i be integrable. This is meaningful if $k > 2$.

(3.5). Conversely let us assume Q_i to be integrable. Then, under our assumptions, one has at the same time

$$p_r(TM) = 0 \qquad p_r(Q_i) = 0 \qquad 2r > \max(m, 4q_i)$$

hence an account of (3.2)

$$p_h(Q_i) P_s(E_i) = 0 \qquad \forall h, s \geq 1, h + s = r.$$

This ends the proof of theorem B.

Remark. The results of theorem (2.1) hold in the more general situation of an almost multifoliated riemannian structures on a manifold, i.e.

$$TM = E_1 + \cdots + E_k$$

where E_i are not necessarily complementary. Infact by increasing the number of distributions, with a suitable choice of the metric [7] it is possible go back to the previous situation.

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