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Arrigo Cellina

On the differential inclusion $x' \in [-1, +1]$

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *On the differential inclusion* $x' \in [-1, +1]$. Nota di ARRIGO CELLINA (*), presentata (**) dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Si mostra che l'insieme delle soluzioni dell'inclusione $x' \in \{-1, +1\}$ è un sottoinsieme della seconda categoria dell'insieme delle soluzioni di $x' \in [-1, +1]$.

INTRODUCTION

It is a well established fact that, under fairly general conditions, the set of solutions to

$$x' \in F(x) \quad , \quad x(0) = 0$$

is a dense subset (in the topology of C (I)) of the set of solutions to

$$x' \in \overline{co}\, F(x) \quad , \quad x(0) = 0 .$$

The present note studies a very particular case, i.e. the (scalar) differential inclusion $x' \in \{-1, +1\}$ and its relations with its convexified analogue, $x' \in [-1, +1]$. We shall denote by K the set of solutions to this last inclusion with the initial condition $x(0) = 0$: it is a closed subset of C(I), hence a complete metric space.

The set K^0, the solutions to $x' \in \{-1, +1\}$, is a dense subset of it. The point of this note is to show that K^0 is a subset of K of the second cathegory.

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(*) Nella seduta del 26 giugno 1980.

RESULTS

THEOREM 1. *The set* K^0 *is a* $\mathcal{G}_\delta$*-dense subset of* K.

The proof of Theorem 1 is an easy consequence of the following Lemma, of some interest in itself. Let x be an absolutely continuous function whose derivate has a.e. values in $[-1, +1]$. Set $\rho_x(t) = d(x'(t), \{-1, +1\}) = \inf\{1 - x'(t), x'(t) + 1\}$. It is a measurable function with values a.e. in $[0, 1]$. Then we have

LEMMA. *Let* x *and* x_n *belong to* K, $x_n \to x$. *Then*

$$\limsup \int_I \rho_{x_n} \leq \int_I \rho_x .$$

Before proceeding to the proof of the Lemma, let us recall the following definitions and results from measure theory.

A point t belonging to a measurable subset E or R is called *point of density* if

$$\lim \frac{m(I_n \cap E)}{m(I_n)} = 1$$

for I_n the interval $(t - 1/n, t + 1/n)$. It is known that almost every $t \in E$ is a point of density.

Let $\mathcal{T}$ be a collection of closed intervals with the property that for every $\varepsilon > 0$ and $t \in E$ there exists an element $I \in \mathcal{T}$ such that $t \in I$ and $m(I) < \varepsilon$. Then $\mathcal{T}$ is called a covering of E *in the sense of Vitali*. Then the following result holds for such a covering $\mathcal{T}$: there exists a countable subcollection $\{I_m\}$ of $\mathcal{T}$ such that $m(E \setminus (\cup I_m)) = 0$ and the I_m are pairwise disjoint [1], [2].

Proof of the Lemma. The proof proceeds into two steps. First we estimate

$$\int_E \rho_y - \int_E \rho_x$$

on a (measurable) E where x' is of constant sign and that differs by little from being an interval. Second we reduce the general case to this estimate by the Vitali covering theorem.

Ad *i*). Let $J \subset I$ be an interval, E a measurable subset of J such that $x' \geq 0$ on E and such that $m(E) \geq m(J)(1 - \sigma)$. Let $y \in K$ and η be such that

$$\left| \int_J (y' - x') \right| < \eta .$$

We claim that

(1) $$\int_E \rho_y \leq \int_E \rho_x + \eta + 2\sigma m(J).$$

In fact set $A^+ = \{t \in E \mid y'(t) \geq x'(t)\}$, $A^- = \{t \in E \mid y'(t) < x'(t)\}$. $E = A^+ \cup A^-$. On A^+,

$$\rho_y(t) = 1 - y'(t)$$

while on A^-

$$\rho_y(t) = \begin{cases} |-1 - y'| = |1 + y'| = |1 - |y'||, & \text{when } y'(t) \leq 0 \\ |1 - y'| = |1 - |y'||, & \text{when } y'(t) > 0. \end{cases}$$

Then

$$\int_{A^+} \rho_y = \int_{A^+} (1 - y') = \int_{A^+} (1 - x') + \int_{A^+} (x' - y')$$

and

$$\int_{A^-} \rho_y = \int_{A^-} (1 - |y'|) \leq \int_{A^-} (1 - y') = \int_{A^-} (1 - x') + \int_{A^-} (x' - y')$$

i.e.

(2) $$\int_E \rho_y \leq \int_E (1 - x') + \int_E (x' - y').$$

On the other hand

$$\int_E (x' - y') + \int_{J \setminus E} (x' - y') = \int_J (x' - y') \leq \left| \int_J (x' - y') \right| < \eta,$$

hence

$$\int_E (x' - y') \leq \eta - \int_{J \setminus E} (x' - y') \leq \eta + 2\sigma m(J).$$

Our assumption on x' implies that $1 - x' = \rho_x$, i.e. that (2) is the seeked inequality (1). Remark also that (1) would still hold were we to assume $x' > 0$ on E, or $x' \leq 0$ or $x' < 0$ on E.

Ad *ii*). Denote by $I(t, \delta)$ the interval $[t - \delta, t + \delta]$. Consider the interval I and set $E^+ = \{t \in I \mid x'(t) \geq 0\}$, $E^- = \{t \in I \mid x'(t) < 0\}$. The sets E^+ and E^- are measurable. Consider E^+. Almost every point of it is of density: set E_D^+ to be the set of points of density of E^+. Fix $\varepsilon > 0$. For every $t \in E_D^+$ there is $\delta(t) > 0$ such that $\delta \leq \delta(t)$ implies

$$\frac{m(E^+ \cap I(t,\delta))}{2\delta} > 1 - \frac{\varepsilon}{8m(I) + \delta} .$$

The collection

$$\left\{ I(t,\delta) \right\}_{t \in E_D^+}^{\delta \leq \delta(t)}$$

is a Vitali covering of E_D^+. Hence there exists M^+, a null set, and a countable subcovering of $E_D^+ \setminus M^+$

$$\{I_n\} = I(t_n, \delta_n)$$

by disjoint intervals.

Let ν be such that $\sum_{n=\nu+1}^{\infty} 2\delta_n \leq \varepsilon m(E^+)/2m(I)$. Consider $\{I_n\}_{n=1}^{\nu}$ and set

$$\delta^+ = \min\{\delta_n : n = 1, \cdots, \nu\}$$

Also, set η^+ to be $\eta^+ = (2\varepsilon\delta^+)/(2m(I) + \varepsilon)$.

Let $y \in K$ be such that $\|x - y\| \leq \eta^+$. We claim that

$$\int_I \rho_y \leq \int_I \rho_x + \varepsilon .$$

Consider any I_m, $1 \leq m \leq \nu$. Then

$$\left| \int_{I_m} (x' - y') \right| = | y_m(t_m + \delta_m) - x(t_m + \delta_m) - (y_m(t_m - \delta_m) - x(t_m - \delta_m)) |$$

and

$$\frac{m(E^+ \cap I_m)}{2\delta_m} > 1 - \varepsilon/(8m(I) + \varepsilon) .$$

Set $\sigma(\varepsilon) = \varepsilon/(8m(I) + \varepsilon)$. Then $1 - \sigma \leq m(E^+ \cap I_m)/m(I_m)$ implies

$$\sigma m(I_m) \leq \sigma/(1 - \sigma) m(E^+ \cap I_m) = (\varepsilon/8m(I)) m(E^+ \cap I_m) .$$

Also, by our choice of η^+,

$$\eta^+ = 2\delta^+ \sigma(\varepsilon) \leq 2\sigma(\varepsilon) m(I_m) \leq \varepsilon m(E^+ \cap I_m)/4 .$$

Applying to $E^+ \cap I_m$ the estimate of point i) we have

$$\int_{E^+ \cap I_m} \rho_y \leq \int_{E^+ \cap I_m} \rho_x + \eta^+ + 2m(I_m)\sigma(\varepsilon) \leq \int_{E^+ \cap I_m} \rho_x + (\varepsilon/4 + \varepsilon/4) \frac{m(E^+ \cap I_m)}{m(I)} .$$

Taking unions over m, $1 \leq m \leq \nu$, and recalling that the I_m are disjoint, $\cup(E^+ \cap I_m) = E^+ \cap (\cup I_m)$ and

$$\int_{\cup(E^+ \cap I_m)} \rho_y \leq \int_{\cup(E^+ \cap I_m)} \rho_x + \varepsilon/2 \; m(E^+ \cap (\cup I_m))/m(I) .$$

Finally, since $E^+ = M^+ \cup (E^+ \cap (\cup I_m)) \cup E^+ \setminus (E \cap (\cup I_m))$ and from our choice of ν and the above estimate,

$$\int_{E^+} \rho_y \leq \int_{E^+ \cap (\cup I_m)} \rho_y + (\varepsilon m(E^+))/(2\, m(I))$$

$$\leq \int_{E^+ \cap (\cup I_m)} \rho_x + \varepsilon/2 \; m(E^+ \cap (\cup I_m))/m(I) + (\varepsilon m(E^+))/(2\, m(I))$$

$$\leq \int_{E^+} \rho_x + \varepsilon/2 \; m(E^+)/m(I) + \varepsilon/2 \; m(E^+)/m(I)$$

so that

$$\int_{E^+} \rho_y \leq \int_{E} \rho_x + \varepsilon \frac{m(E^+)}{m(I)} .$$

The same reasoning on E^- yields a η^- such that $y \in K$, $\| y - x \| < \eta^-$ implies

$$\int_{E^-} \rho_y \leq \int_{E} \rho_x + \varepsilon \frac{m(E^-)}{m(I)} .$$

Adding the above inequalities we obtain, for $\eta = \min\{\eta^+, \eta^-\}$

$$y \in K , (\| y - x \| < \eta) \Rightarrow \int_E \rho_y \leq \int_E \rho_x + \varepsilon .$$

This proves our claim *ii*) and the Lemma.

Proof of Theorem 1. For any $\varepsilon > 0$ consider the set

$$H_\varepsilon = \left\{ x \in K \;\middle|\; \int_I \rho_x \geq \varepsilon \right\} .$$

Since for $x \in K^0$, $\rho_x = 0$, H_ε has empty intersection with K^0. The above Lemma implies that H_ε is closed; in fact $x_n \to x$, $\int \rho_{x_n} \geq \varepsilon$ imply that

$$\int_I \rho_x \geq \limsup \int_I \rho_{x_n} \geq \varepsilon .$$

Moreover the interior of H_ε must be empty: were H_ε to contain an open set, it would contain a point of K^0, since this set is dense. Thus the complement of H_ε is open and dense, and contains K^0. Let ε_n be any positive sequence converging to zero. The intersection of the complements to H_{ε_n} is, by Baire's Theorem, a $\mathcal{G}_\delta$ dense subset, containing K^0. This intersection is, actually, K^0: any $\tilde{x}$ not in K^0 would have $\rho_{\tilde{x}}(t)$ positive on a set of positive measure, hence for some $\varepsilon_n > 0$, it would belong to H_{ε_n}. Hence K^0 is a $\mathcal{G}_\delta$ dense subset of K.

NOTE ADDED IN PROOF. The function ρ is concave, i.e.$-\rho$ is convex. A result much more general than our Lemma is in: C. OLECH, *The weak lower semicontinuity of integral functionals* « J. Opt. Th. App. », *19* (1976), 3-16.

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[1] C. W. BURRILL and J. R. KNUDSEN (1969) - *Real Variables*, Holt Reinehart and Winston, New York.

[2] G. VITALI and G. SANSONE (1943) - *Moderna Teoria delle Funzioni di Variabile Reale*, Zanichelli, Bologna.