Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A stability criterion for the equilibrium of a thermoelastic dielectric in the presence of conductors

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **68** (1980), n.6, p. 526–532.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1980_8_68_6_526_0>

Fisica matematica. — *A stability criterion for the equilibrium of a thermoelastic dielectric in the presence of conductors* (*). Nota di DONATELLA IANNÉCE, presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — Si prova un teorema di stabilità per l'equilibrio di un dielettrico termoelastico non lineare il quale risulti in contatto con conduttori su tutta la sua superficie. Si prova, inoltre, per un dielettrico lineare, un teorema di esistenza, unicità e stabilità per una configurazione di equilibrio prossima a quella naturale.

1. INTRODUCTION

In a previous paper [1] I proved in particular that for a thermoelastic dielectric $\mathcal{S}$, in the presence of conductors at *constant potential*, there exists a functional V (depending on deformation, temperature and electric field) which is connected to the *total enthalpy* of $\mathcal{S}$. Such a functional V verifies the condiction $\dot{V} \leq 0$ in every motion of $\mathcal{S}$ as a consequence of the principles of Thermodynamics. Moreover, I proved that every equilibrium configuration $\mathcal{C}_e$, which corresponds to a minimum for the total enthalpy of $\mathcal{S}$, is Liapounov stable with respect to the measures V(0) and V(t). This result represents an extention to the case of thermoelastic dielectrics of the energy criterion already extended to the thermoelastic bodies by M. Gurtin [2], [3].

In this paper, the Liapounov stability estimates for an equilibrium configuration $\mathcal{C}_e$ of $\mathcal{S}$ are improved *both for the* cases of *constant potential* and *constant charge*. More precisely, by employing the variational formulation of the equilibrium problem which A. Romano and I found in [4], I show (section 2) that when the second Fréchet differential of functional defined by (2.4) of this paper is strongly coercive, the Liapounov stability of $\mathcal{C}_e$ with respect to the mean squares of electric field, displacement gradient, velocity and temperature differences is assured.

In section 3 I show this stability conditions to imply also the existence and the uniqueness of the natural equilibrium of *linear* dielectrics. Moreover, these conditions are realized when the specific enthalpy ζ of $\mathcal{S}$ is definite positive.

(*) Work supported by G.N.F.M. of Italian C.N.R.
(**) Nella seduta del 26 giugno 1980.

2. A LIAPOUNOV STABILITY THEOREM FOR THE EQUILIBRIUM OF A THERMOELASTIC DIELECTRIC.

Let $\mathscr{S}$ be a thermoelastic dielectric, If $\mathscr{C}_*$ and $\mathscr{C}_e$ represent respectively the reference configuration and the equilibrium one, we shall denote by (1) X^L and x_e^i the Lagrangian and Eulerian coordinates of a particles $\mathbf{X} \in \mathscr{S}$. Consequently the functions

$$x^i = x_e^i(X^L)$$

represent the deformation $\mathscr{C}_* \to \mathscr{C}_e$ and $F_L^i = x_{e,L}^i$ the deformation gradient. I employ also the following notations: $\boldsymbol{u}_e = \boldsymbol{x}_e - \mathbf{X}$ is the displacement field, θ_e the absolute equilibrium temperature, $\mathbf{T}_{*e}$ the Piola–Kirchhoff stress tensor, $\boldsymbol{n}_*$ the exterior unit normal vector to the surface element $d\sigma_*$ of $\partial\mathscr{C}_*$, φ_e the electric potential, $\mathscr{D}_e = J_e \mathbf{F}_e^{-1} \mathbf{D}_e$ the lagrangian induction field, where $\mathbf{D}_e$ is the Eulerian induction field at equilibrium and $J_e = \det \mathbf{F}_e$.

Moreover, I shall suppose that $\mathbf{T}_{*e}$ and $\mathscr{D}_e$ are functions of Grad $\boldsymbol{u}_e$, Grad φ_e and θ_e (2).

In this paper the stability of the following problem of equilibrium of $\mathscr{S}$ is considered. Let $\mathscr{S}$ be in contact with conductors on the whole boundary. On the part $\partial\mathscr{C}'_* \subset \partial\mathscr{C}_*$ act external forces, whose surface density is t_*, while the remaining part is fixed; finally, on $\partial\mathscr{C}''_* \subset \partial\mathscr{C}_*$ and $\partial\mathscr{C}_* - \partial\mathscr{C}''_*$ the total charge Q_0 and the potential $\varphi_e = 0$ are respectively assigned.

This problem of equilibrium is expressed by the system:

$$(2.1)\qquad \begin{aligned} &\operatorname{Div} \mathbf{T}_{*e}(\operatorname{Grad} \boldsymbol{u}_e, \operatorname{Grad} \varphi_e, \theta_e) = \mathbf{0},\\ &\operatorname{Div} \mathscr{D}_e(\operatorname{Grad} \boldsymbol{u}_e, \operatorname{Grad} \varphi_e, \theta_e) = 0 \end{aligned}$$

with the boundary conditions:

$$(2.2)\qquad \begin{aligned} &\mathbf{T}_{*e}\cdot \boldsymbol{n}_* = \boldsymbol{t}_*, &&\text{on } \partial\mathscr{C}'_*\\ &\boldsymbol{u}_e = \mathbf{0}, &&\text{on } \partial\mathscr{C}_* - \partial\mathscr{C}'_*\\ &\int_{\partial\mathscr{C}''_*} \mathscr{D}_e\cdot\boldsymbol{n} = Q_0 &&\\ &\varphi_e = 0 &&\text{on } \partial\mathscr{C}_* - \partial\mathscr{C}''_*. \end{aligned}$$

(1) The Latin small letters denote Eulerian indexes while tha Latin capital letters denote material indexes. Moreover, $a_{,L}$ denotes the derivative of a with respect to X^L.

(2) Grad, Div. and Rot refer to differentiation with respect to coordinates X^L.

It is worthwhile to observe that, in the considered case, the conductors completely delimitate $\mathcal{S}$ so that, vanishing the electric field in these conductors, $\boldsymbol{t}_*$ is only a mechanical force.

In [4], A. Romano and I proved that the solution of the problem (2.1), (2.2) are extremals of the functional:

$$(2.3) \qquad \mathcal{F}(\boldsymbol{u}, \theta_e, \varphi) = \int_{\mathcal{C}_*} \rho_* \zeta(u^i_{,L} \,|\, \theta_e \,|\, \varphi_{,L}) \, d\mathcal{C}_* + \int_{\partial\mathcal{C}'_*} \boldsymbol{t}_* \cdot \boldsymbol{u} \, d\sigma + Q_0 \, \varphi_{\partial\mathcal{C}''_*}$$

where ζ is the specific enthalpy of $\mathcal{S}$.

These extremals have to be found in the Sobolev space $\mathcal{U} = (W^1_0(\mathcal{C}_*))^4$ of functions $\boldsymbol{h} = (h^\alpha(\cdot)) = (v^i(\cdot), \lambda(\cdot))$ from $\mathcal{C}_*$ into $\mathbf{R}^4$, which are square sommable with their first derivatives and satisfy the conditions:

$$v^i(\mathbf{X}) = 0 \qquad \text{on } \partial\mathcal{C}_* - \partial\mathcal{C}'_*$$

$$\lambda(\mathbf{X}) = 0 \qquad \text{on } \partial\mathcal{C}_* - \partial\mathcal{C}''_*$$

$$\lambda(\mathbf{X}) = \text{const} \qquad \text{on } \partial\mathcal{C}''_* .$$

$\mathcal{U}$ becomes an Hilbert space when we introduce into it the scalar product:

$$(2.4) \qquad (\boldsymbol{h}, \boldsymbol{k})_{\mathcal{U}} = \sum_1^4{}_\alpha \left(\int_{\mathcal{C}_*} h^\alpha k^\alpha \, d\mathcal{C}_* + \sum_1^3{}_L \int h^\alpha_{,L} k^\alpha_{,L} \, d\mathcal{C}_* \right) .$$

In order to prove a stability theorem for the aforesaid equilibrium problem, I shall employ the thermodynamical inequality which in [1] I derived in every motion of $\mathcal{S}$ from the balance of energy and the Clausius–Duhem inequality. In the Lagrangian form it results:

$$(2.5) \qquad \frac{d}{dt} \int_{\mathcal{C}_*} \rho_* \left[\zeta(u^i_{,L} \,|\, \theta_e \,|\, \varphi_{,L}) + \frac{k}{2} (\theta - \theta_e)^2 + \frac{1}{2} \dot{x}^2 \right] d\mathcal{C}_* \leq \int_{\partial\mathcal{C}_*} (\dot{\boldsymbol{x}} \cdot \mathbf{T}_* - \mathcal{D} \, \dot{\varphi}) \cdot \boldsymbol{n}_* \, d\sigma_*$$

where a dot denotes a temporal derivative, $k = - \left[\dfrac{\partial^2 \zeta}{\partial \theta^2} \right]_{\theta_e}$ is positive, because the specific heat c is positive (see [1]) and $\theta \neq \theta_e$ is the temperature field during the motion.

Into (2.5) the term of free energy of vacuum, which appears in (3.7) of [1], has been omitted because in this paper the system $\mathcal{S}$ reduces only to

(3) I recall that equation (2.1) is one of the Maxwell equations expressed in Lagrangian form which A. Romano and I found in [4].

the dielectric. The integral on the right hand side of (2.5), when we take into account the boundary conditions (2.2), becomes

$$\int_{\partial\mathscr{C}'_*} \dot{\boldsymbol{x}} \cdot \boldsymbol{t}_* \, d\sigma_* - Q_0 \, \dot{\varphi}_{\partial\mathscr{C}''} \; . \tag{2.6}$$

Moreover, if the external surface forces are supposed weakly conservative, i.e.

$$\int_{\partial\mathscr{C}_*} \dot{\boldsymbol{x}} \cdot \boldsymbol{t}_* \, d\sigma_* \equiv - \frac{dU_m}{dt}$$ (4)

we have

$$\int_{\partial\mathscr{C}_*} (\dot{\boldsymbol{x}} \cdot \mathbf{T}_* - \mathscr{D}\dot{\varphi}) \cdot \boldsymbol{n}_* \, d\sigma_* = - \frac{dU}{dt} \tag{2.7}$$

where

$$U = Q_0 \, \varphi_{\partial\mathscr{C}''_*} + U_m \, . \tag{2.8}$$

By adding to (2.5) the term

$$\frac{d}{dt} \int_{\mathscr{C}_*} \rho_* \, \zeta \, (u^i_{e,L} \mid \theta_e \mid \varphi_{e,L}) \, d\mathscr{C}_* = 0 \, ,$$

in view of (2.7), we obtain the following inequality

$$\frac{d}{dt} V(t) \equiv \frac{d}{dt} \left(\int_{\mathscr{C}_*} \rho_* \left[\zeta \, (u^i_{,L} \mid \theta_e \mid \varphi_{,L}) - \zeta \, (u^i_{e,L} \mid \theta_e \mid \varphi_{e,L}) \right] \right) d\mathscr{C}_* + \int_{\mathscr{C}_*} \rho_* \left[\frac{k}{2} (\theta - \theta_e)^2 + \frac{1}{2} \dot{x}^2 \right] d\mathscr{C}_* + U \Big) \leq 0 \, , \tag{2.9}$$

which suggests to employ $V(t)$, defined by (2.9), as Liapounov function.

In concluding these preliminary considerations, it is worthwhile to observe that functional (2.4) assumes the form

$$\mathscr{F} = \int_{\mathscr{C}_*} \rho_* \, \zeta \, (u^i_{,L} \mid \theta_e \mid \varphi_{,L}) \, d\mathscr{C}_* + U$$

when the force acting on $\mathscr{S}$ are weakly conservative.

(4) In [5] we can find the expressions of U_m when $\partial\mathscr{C}_*$ is fixed, either $\mathscr{S}$ is subject on $\partial\mathscr{C}'_*$ to a constant and uniform pressure, or to a dead load, to elastic forces and so on.

Now it is possible to prove the following Liapounov stability theorem for the equilibrium in the presence of weakly conservative forces:

THEOREM. *If* U *does not explicitly depend on time and the functional*

$$\mathscr{H} = \int_{\mathscr{C}_*} \rho_* \left(\zeta (u^i_{,L} \mid \theta_e \mid \varphi_{,L}) - \zeta (u^i_{e,L} \mid \theta_e \mid \varphi_{e,L}) \right) d\mathscr{C}_* + U \tag{2.10}$$

verifies the condition

$$d^2 \mathscr{H} (\boldsymbol{h}_e \mid \boldsymbol{k}, \boldsymbol{k}) \geq a \| \boldsymbol{k} \|^2_{\mathscr{U}}, \qquad a > 0 \tag{2.11}$$

then the equilibrium configuration $\boldsymbol{h}_e = (\boldsymbol{u}_e, \theta_e, \varphi_e)$ *is stable with respect to the measures*

$$(2.12) \qquad \begin{aligned} \rho_0 &\equiv V(0), \\ \rho_t &\equiv \| \boldsymbol{h} \|_{\mathscr{U}} + \int_{\mathscr{C}_*} \tfrac{1}{2} \rho_* (k (\theta - \theta_e)^2 + \dot{x}^2) \, d\mathscr{C}_* \end{aligned}$$

Proof. Function $V(t)$ being decreasing in every process of $\mathscr{S}$, we have $V(0) \geq V(t)$ and the first condition of the Liapounov stability criterion [(5)] is satisfied. On the other hand, (2.10) and (2.11) imply

$$V = \mathscr{H} + \frac{1}{2} \int_{\mathscr{C}_*} \rho_* [k (\theta - \theta_e)^2 + \dot{x}^2] \, d\mathscr{C}_*$$

$$\mathscr{H} (\boldsymbol{h}, \theta_e) - \mathscr{H} (\boldsymbol{h}_e, \theta_e) \geq d^2 \mathscr{H} (\boldsymbol{h}_e \mid \boldsymbol{k}, \boldsymbol{k}) \geq a \| \boldsymbol{k} \|^2_{\mathscr{U}}$$

and also the second condition of Liapounov stability criterion is verified. Therefore, the stability of $\mathscr{C}_e$ with respect to the measures (2.12) is assured.

This theorem allows to obtain stability extimates better than the usual ones. In fact for thermoelastic materials [3], [4] and elastic dielectrics [1], stability theorems are proved by employing the ordinary Liapounov measures to evaluate the perturbations. Of course, it is quite evident the advantage of adopting the Sobolev norms as measures of stability.

3. AN EXISTENCE, UNIQUENESS AND STABILITY THEOREM FOR A LINEAR ELASTIC DIELETRIC.

Let us assume a natural configuration $\mathscr{C}_*$ as reference configuration. If boundary data (2.2) are suitable small, we can suppose the corresponding equilibrium configuration $\mathscr{C}_e$ so near to $\mathscr{C}_*$ that it is possible to adopt the linear

(5) See, Knops and Wilkes [5].

approximation. By using again the notations of the previous section, this approximation in particular implies:

$$\rho_* \zeta(\boldsymbol{h}) = \tfrac{1}{2} \sum_{i,j,L,M}^{3} G_{ijLM}\, h_{i,L}\, h_{j,M}\,, \tag{3.1}$$

where $\zeta(o) = o$ and G_{ijLM} -*s* are constant.

On the other hand, recalling [4] and the results of section 2, the equilibrium solution $\boldsymbol{h}_e$ is an extremal of functional (2.3), which in our approximations can be written:

$$\mathcal{F}(\boldsymbol{h}) = \sum_{ijLM=1}^{3} \int_{\mathcal{C}_*} G_{ijLM}\, h_{i,L}\, h_{j,M}\, d\mathcal{C}_* + \int_{\partial\mathcal{C}'_*} \boldsymbol{t}_* \cdot \boldsymbol{u}\, d\sigma_* + Q_0\, \varphi_{\partial\mathcal{C}''_*}(\boldsymbol{h} \equiv (\boldsymbol{u}, \varphi)). \tag{3.2}$$

Therefore, $\boldsymbol{h}_e$ is a solution of the problem (2.1)–(2.2) if it satisfies the equation

$$d\mathcal{F}(\boldsymbol{h}_e \mid \boldsymbol{k}) = o\,, \qquad \forall \boldsymbol{k} \in \mathcal{U}\,. \tag{3.3}$$

This equation, when we introduce the linear operators

$$\begin{aligned} A(\boldsymbol{h}_e, \boldsymbol{k}) &= \sum_{ijLM=1}^{3} \int_{\mathcal{C}_*} G_{ijLM}\, h^{i}_{e,L}\, k_{i,M}\, d\mathcal{C}_*\,, \qquad \forall \boldsymbol{k} \in \mathcal{U} \\ F(\boldsymbol{k}) &= \int_{\partial\mathcal{C}'_*} \boldsymbol{t}_* \cdot \boldsymbol{u}\, d\sigma_* + Q_0\, \varphi_{\partial\mathcal{C}''_*}\,, \end{aligned} \tag{3.4}$$

assumes the form

$$A(\boldsymbol{h}_e, \boldsymbol{k}) = F(\boldsymbol{k})\,. \tag{3.5}$$

We can now prove in the linear approximation the

THEOREM. *If the quadratic from* (3.1) *is definite positive and* $t_{*i} \in L_2(\partial\mathcal{C}''_*)$, *then there exists one and only one solution of the problem* (2.1)–(2.2) *which is stable with respect to the measures* (2.12).

Proof. It is easy to derive in these hypotheses the continuity of the operators A and F by application of the Schwartz inequality and moreover to desume that A is strongly coercive by the Poincarè inequality (see [5], p. 207). Then both the hypotheses of the Lax–Milgran theorem are satisfied and consequently the existence and uniqueness of the solution of the problem (2.1)–(2.2) is assured.

On the other hand, the strongly coerciveness of A (and consequently of $\mathcal{F}$ and $\mathcal{H}$) assures that (3.11) si satisfied. The theorem is proved as a consequence of the previous one.

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