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On a physical interpretation of Fichera's function

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Fisica matematica. — *On a physical interpretation of Fichera's function.* Nota di GEORGE JAIANI, presentata (*) dal Socio G. FICHERA.

RIASSUNTO. — Viene interpretato, dal punto di vista della teoria delle volte prismatiche sottili di I. N. Vekua, il significato fisico di una funzione impiegata da G. Fichera nella teoria delle equazioni lineari alle derivate parziali del secondo ordine con forma caratteristica non negativa.

I. N. Vekua [4], [5] set the problem concerning the investigation of prismatic shells with cusped edges. This problem is connected with the consideration of degenerate equations of higher order or systems of degenerate elliptic equations of second order.

1. SHELLS' CUSPS

Let $Ox_1 x_2 x_3$ be a Cartesian coordinate system. We denote by D the projection of the prismatic shell on the plane $Ox_1 x_2$. Let L be a piecewise-smooth boundary of the domain D and $H(x_1, x_2) = 0$ its equation in some neighbourhood of a smooth point P of L. Hence

$$\frac{\partial H(x_1, x_2)}{\partial x_\alpha}, \quad \alpha = 1, 2$$

are continuous functions in some neighbourhood of the above mentioned smooth point P and

$$\left[\frac{\partial H(x_1, x_2)}{\partial x_1}\right]^2 + \left[\frac{\partial H(x_1, x_2)}{\partial x_2}\right]^2 \neq 0$$

at this point. Further we assume that grad H has the direction of the inside normal $\boldsymbol{n}$.

By

$$2h(x_1, x_2) = \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) \geq 0,$$

where $\overset{(+)}{h}(x_1, x_2)$ and $\overset{(-)}{h}(x_1, x_2)$ are, respectively, the upper and the lower sur faces of the shell [4], we denote the shell thickness.

DEFINITION 1. The points $P \in L$ at which the thickness vanishes will be called cusps, while the other points will be called regular ones.

(*) Nella seduta del 10 maggio 1980.

Let $\omega(P)$ be a neighbourhood of P and let the functions $\overset{(+)}{h}, \overset{(-)}{h}$ have continuous derivatives in $\omega(P) \cap D \cup L$, nevertheless the continuity of these derivatives could fail in P or on some arc of L containing P, where the derivatives will be infinite.

DEFINITION 2. A cusp which is also a smooth point of the boundary L will be called:

a blunt cusp if

$$\lim_{Q \to P} \frac{\partial h(Q)}{\partial n} = +\infty, \qquad Q \in \omega(P) \cap D;$$

a sharpened cusp if

$$+\infty > \lim_{Q \to P} \frac{\partial h(Q)}{\partial n} \geq 0, \qquad Q \in \omega(P) \cap D.$$

In figures 1–6 are represented all the possible configurations of the normal sections (side views) of shells at the point P. For simplicity, we suppose $P \equiv O$. $\overset{(+)}{T}$ and $\overset{(-)}{T}$ denote the tangents at the point P, respectively to the sections $\overset{(+)}{h}(n)$ and $\overset{(-)}{h}(n)$ of the surfaces $\overset{(+)}{h}$ and $\overset{(-)}{h}$, with the plane which passes through the x_3-axis and contains the vector $\boldsymbol{n}$.

I. $\dfrac{\partial h(P)}{\partial n} = +\infty$.

In this case it can be:

a) $\dfrac{\partial \overset{(+)}{h}(P)}{\partial n} = +\infty, \quad \dfrac{\partial \overset{(-)}{h}(P)}{\partial n} = -\infty$ (see fig. 1);

b) $\dfrac{\partial \overset{(+)}{h}(P)}{\partial n} = +\infty, \quad -\infty < \dfrac{\partial \overset{(-)}{h}(P)}{\partial n} < 0$ (see fig. 2);

c) $\dfrac{\partial \overset{(+)}{h}(P)}{\partial n} = +\infty, \quad \dfrac{\partial \overset{(-)}{h}(P)}{\partial n} = 0$ (see fig. 3).

Other two configurations are possible and they can be obtained by *b*) and *c*) interchanging the roles of $\overset{(+)}{h}$ and $\overset{(-)}{h}$.

II. $+\infty > \dfrac{\partial h(P)}{\partial n} > 0$.

In this case it can be:

d) $+\infty > \dfrac{\partial \overset{(+)}{h}(P)}{\partial n} > 0, \quad -\infty < \dfrac{\partial \overset{(-)}{h}(P)}{\partial n} < 0$ (see fig. 4);

e) $+\infty > \dfrac{\partial \overset{(+)}{h}(P)}{\partial n} > 0, \quad \dfrac{\partial \overset{(-)}{h}(P)}{\partial n} = 0$ (see fig. 5).

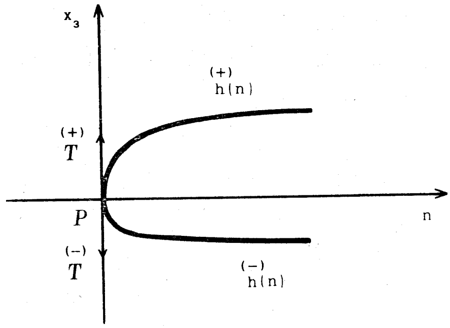

fig. 1

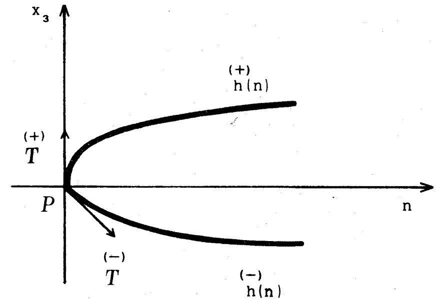

fig. 2

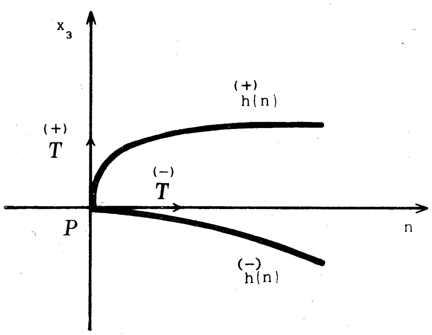

fig. 3

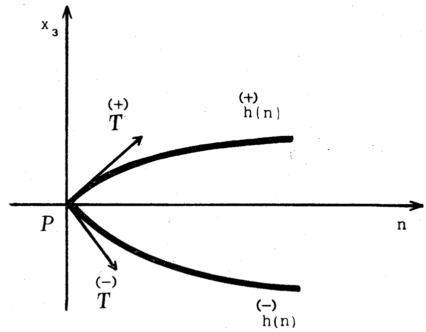

fig. 4

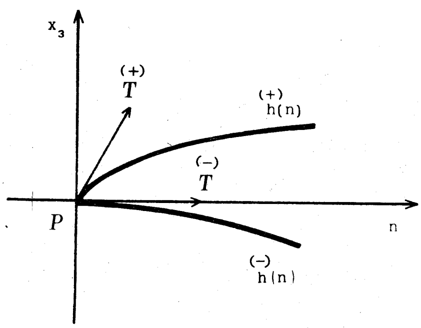

fig. 5

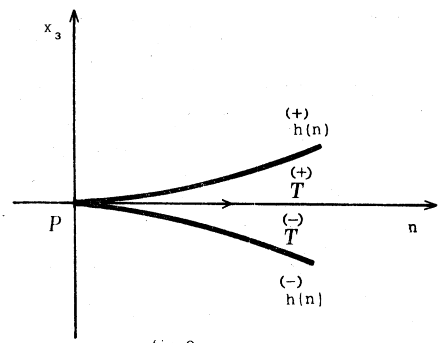

fig. 6

Another configuration is possible and it can be obtained by *e*) interchanging the roles of $\overset{(+)}{h}$ and $\overset{(-)}{h}$.

III. $\dfrac{\partial h(\mathrm{P})}{\partial n} = 0$.

In this case

f) $\dfrac{\partial \overset{(+)}{h}(\mathrm{P})}{\partial n} = 0 \quad , \quad \dfrac{\partial \overset{(-)}{h}(\mathrm{P})}{\partial n} = 0$ (see fig. 6).

CONCLUSIONS

In the case of the sharpened cusps the section of the shell touchs the x_3-axis at the point P with the vertex, while in the case of blunt cusps the section of the shell is tangent to x_3-axis at the point P.

The angle of a blunt cusp (i.e. the angle between $\overset{(+)}{T}$ and $\overset{(-)}{T}$) belongs to the closed interval $[\pi/2, \pi]$, the angle of a sharpened cusp belongs to the interval $[0, \pi/2)$ (except the case *d*) in which the angle belongs to the open interval $(0, \pi)$.

In the case of blunt cusps at least one of the curves $\overset{(+)}{h}(n), \overset{(-)}{h}(n)$ is perpendicular, at the point P, to the middle plane of the prismatic shell.

If $\overset{(+)}{h} = -\overset{(-)}{h} = h$ we have only the cases of type *a*), *d*), *f*) and hence the angle of the cusp is equal to π for a blunt cusp and is less than π for a sharpened cusp.

2. VEKUA'S EQUATIONS OF THIN SHELLS

By approximation of order $N = 0$ of Vekua's variant of the thin shell theory we mean [4], [5]:

$$u_i(x_1, x_2, x_3) \approx \frac{1}{2h(x_1, x_2)} \int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} u_i(x_1, x_2, x_3)\, dx_3 = v_i(x_1, x_2), \qquad i = 1, 2, 3,$$

where u_i is the displacement's component respect to the x_i-axis $(i = 1, 2, 3)$.

The basic system of equations has the form

$$(1) \qquad \mu h \Delta v_\beta + (\lambda + \mu) h \frac{\partial \Theta(x_1, x_2)}{\partial x_\beta} + \mu \frac{\partial h}{\partial x_\alpha} \frac{\partial v_\alpha}{\partial x_\beta} + \mu \frac{\partial h}{\partial x_\alpha} \frac{\partial v_\beta}{\partial x_\alpha}$$

$$+ \lambda \frac{\partial h}{\partial x_\beta} \frac{\partial v_\alpha}{\partial x_\alpha} + \frac{1}{2} \overset{\circ}{X}_\beta = 0 \qquad \alpha, \beta = 1, 2, \qquad (x_1, x_2) \in D,$$

$$(2) \qquad h \Delta v_3 + \frac{\partial h}{\partial x_\alpha} \frac{\partial v_3}{\partial x_\alpha} + \frac{1}{2\mu} \overset{\circ}{X}_3 = 0 \quad , \quad (x_1, x_2) \in D,$$

where Δ is the two-dimensional Laplace operator,

$$\Theta(x_1, x_2) = \frac{\partial v_\alpha(x_1, x_2)}{\partial x_\alpha},$$

$$\overset{\circ}{X}_i(x_1, x_2) = \int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} X_i(x_1, x_2, x_3)\, dx_3, \qquad i = 1, 2, 3.$$

By X_i we have denoted the components of the volume force, λ, μ are the Lamè constants and the usual summation convention is used.

All the physical quantities on the shell boundary will be defined by means of their limits calculated from the inside.

3. Fichera's function

Let now consider the equation [1–3]

$$a^{\alpha\beta}(x_1, x_2)\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + b^\alpha(x_1, x_2)\frac{\partial u}{\partial x_\alpha} + c(x_1, x_2)\, u(x_1, x_2) = f(x_1, x_2), \tag{3}$$

$$\alpha, \beta = 1, 2 \quad , \quad (x_1, x_2) \in D,$$

with nonnegative characteristic form, i.e.

$$a^{\alpha\beta}(x_1, x_2)\,\xi_\alpha \xi_\beta \geq 0 \quad , \quad \alpha, \beta = 1, 2 \qquad (x_1, x_2) \in D \cup L, \tag{4}$$

for any real ξ_α, $\alpha = 1, 2$.

Fichera's function is defined in the points of the boundary where

$$a^{\alpha\beta}(x_1, x_2)\, n_\alpha n_\beta = 0$$

and has the following representation:

$$b(x_1, x_2) \equiv \left(b^\alpha - \frac{\partial a^{\alpha\beta}}{\partial x_\beta}\right) n_\alpha, \tag{5}$$

here n_α $(\alpha = 1, 2)$ are the components of the vector $\boldsymbol{n}$.

Let us calculate Fichera's function b^* for the equation which we obtain after having multiplied the equation (3) by a function $\gamma(x_1, x_2)$ which has continuous derivatives in some neighbourhood of the considered point of the boundary. We obtain for b^* the following representation:

$$b^*(x_1, x_2) = \left(\rho b^\alpha - \frac{\partial \rho a^{\alpha\beta}}{\partial x_\beta}\right) n_\alpha = \rho\left(b^\alpha - \frac{\partial a^{\alpha\beta}}{\partial x_\beta}\right) n_\alpha$$

$$- a^{\alpha\beta}\, n_\alpha \frac{\partial \rho}{\partial x_\beta} = \rho b(x_1, x_2),$$

because $a^{\alpha\beta} n_\alpha = 0$. Hence, if the function γ is positive, the sign of Fichera's function does not change.

4. The main statement

Under the below mentioned conditions we prove the following statement: *In a blunt cusp Fichera's function for the equation* (2) *is negative and in a sharpened cusp is non negative.*

In the case of sharpened cusps we have:

$$+\infty > \frac{\partial h(\mathrm{P})}{\partial n} \geq 0\,, \qquad h(x_1, x_2) \in \mathrm{C}^1(\omega(\mathrm{P}) \cap \mathrm{D} \cup \mathrm{L})\ ^{(1)}$$

and from (2), (5) it follows that Fichera's function assumes the form:

$$b(\mathrm{P}) = \left[\frac{\partial h(\mathrm{P})}{\partial x_\alpha} - \frac{\partial h(\mathrm{P})}{\partial x_\alpha}\right] n_\alpha = 0\,.$$

In the case of blunt cusps we have:

$$\frac{\partial h(\mathrm{P})}{\partial n} = +\infty\,, \qquad h(x_1, x_2) \notin \mathrm{C}^1(\omega(\mathrm{P}) \cap \mathrm{D} \cup \mathrm{L})$$

and obviously Fichera's function cannot be defined.

But if it exists a function

$$\rho(x_1, x_2) \in \mathrm{C}^1(\omega(\mathrm{P}) \cap \mathrm{D})\,,$$

which fulfills the following conditions:

1) $\rho(x_1, x_2) > 0$ in $\omega(\mathrm{P}) \cap \mathrm{D} \cup (\mathrm{L} - \tilde{\mathrm{L}})$ (2);

2) $\rho \dfrac{\partial h(x_1, x_2)}{\partial x_\alpha}\,, \quad h \dfrac{\partial \rho(x_1, x_2)}{\partial x_\alpha} \in \mathrm{C}(\omega(\mathrm{P}) \cap \mathrm{D} \cup \mathrm{L})\,, \qquad \alpha = 1, 2\,;$

3) $h(x_1, x_2) \dfrac{\partial \rho(x_1, x_2)}{\partial n}\Bigg|_{(x_1, x_2) = \mathrm{P}} \neq 0\,,$

then

$$\rho(x_1, x_2)\, h(x_1, x_2) \in \mathrm{C}^1(\omega(\mathrm{P}) \cap \mathrm{D} \cup \mathrm{L})\,.$$

By multiplying (3) both sides of the equation (2) by $\rho(x_1, x_2)$ we have:

$$\rho h \Delta v_3 + \rho \frac{\partial h}{\partial x_\alpha} \frac{\partial v_3}{\partial x_\alpha} + \rho \frac{\overset{\circ}{\mathrm{X}}_3}{2\mu} = 0\,, \qquad (x_1, x_2) \in \omega(\mathrm{P}) \cap \mathrm{D}\,.$$

For this equation Fichera's function can be defined and has the following form:

$$b(x_1, x_2) = \left[\rho(x_1, x_2) \frac{\partial h(x_1, x_2)}{\partial x_\alpha} - \frac{\partial \rho(x_1, x_2)\, h(x_1, x_2)}{\partial x_\alpha}\right] n_\alpha =$$

$$= -h(x_1, x_2) \frac{\partial \rho(x_1, x_2)}{\partial n}\,.$$

(1) By C^1 we denote the class of the functions which have continuous derivatives on the set shown in parentheses and by C we denote the class of the continuous functions.

(2) $\tilde{\mathrm{L}}$ is the subset of L where $\partial h / \partial n = +\infty$.

(3) This is permitted because, in the viewpoint of the shell theory, we consider the equation (3) only inside the shell projection and there $\rho > 0$.

Hence

$$b(\mathrm{P}) < 0,$$

because

$$\frac{\partial\rho}{\partial n} \geq 0$$

in $\omega(\mathrm{P}) \cap \mathrm{D}$ for any $\omega(\mathrm{P})$ sufficiently small.

It may be noted that when

$$\frac{\partial h(\mathrm{P})}{\partial n} = 0,$$

it is still possible to introduce a function $\rho(x_1, x_2)$ fulfilling the above mentioned conditions 1)–3) and such that $\rho(\mathrm{R}) = +\infty$, $\mathrm{R} \in \mathring{\mathrm{L}}$ [4]. Then Fichera's function $b(x_1, x_2)$ is positive at the point P, because

$$\frac{\partial\rho}{\partial n} \leq 0$$

in $\omega(\mathrm{P}) \cap \mathrm{D}$ for any $\omega(\mathrm{P})$ sufficiently small.

In particolar, let

$$\frac{\partial h(x_1, x_2)}{\partial n} = O^*(\mathrm{H}^{\varkappa-1}), \quad \text{for } \mathrm{H} \to 0 \text{ [5]}, \qquad 0 < \varkappa < 1, \tag{6}$$

where the notation

$$f = O^*(g) \quad , \quad \mathrm{H} \to 0,$$

means that there exists

$$\lim_{\mathrm{H}\to 0} f \cdot g^{-1} \neq 0, \infty.$$

If the hypothesis (6) is satisfied, then a function $\rho(x_1, x_2)$ can have the following representation:

$$\rho(x_1, x_2) = \int_0^{\mathrm{H}(x_1, x_2)} \frac{d\mathrm{H}}{h}. \tag{7}$$

Let's now verify that the function (7) satisfies the condition 1)–3).

The condition 1) is obviously fulfilled.

(4) $\mathring{\mathrm{L}}$ is the subset of L where $\partial h/\partial n = 0$.

(5) Obviously, from $\mathrm{Q} \to \mathrm{P}$ it follows that $\mathrm{H}(\mathrm{Q}) \to 0$.

We observe first that from (6) it follows:

$$\lim_{H\to 0} \frac{h(x_1, x_2)}{H^{\varkappa}} = \lim_{H\to 0} \frac{\partial h/\partial H}{\varkappa H^{\varkappa-1}} = \frac{1}{\varkappa} \lim_{H\to 0} \frac{\partial h/\partial n\,(\partial H/\partial h)^{-1}}{H^{\varkappa-1}} =$$

$$= \frac{1}{\varkappa} \lim_{H\to 0} \frac{O^*(H^{\varkappa-1})}{H^{\varkappa-1}} \lim_{H\to 0} \left(\frac{\partial H}{\partial n}\right)^{-1} \neq 0, \infty,$$

because

$$\left.\frac{\partial H}{\partial n}\right|_{H=0} = \left[\left(\frac{\partial H}{\partial x_1}\right)^2 + \left(\frac{\partial H}{\partial x_2}\right)^2\right]^{\frac{1}{2}}_{|H=0} \neq 0, \infty.$$

That is

$$h(x_1, x_2) = O^*(H^{\varkappa}) \quad , \quad H \to 0, \qquad 0 < \varkappa < 1.$$

Further:

$$\lim_{H\to 0} \frac{\rho(x_1, x_2)}{H^{1-\varkappa}} = \lim_{H\to 0} \frac{\partial \rho/\partial H}{(1-\varkappa)H^{-\varkappa}} = \frac{1}{1-\varkappa} \lim_{H\to 0} \frac{H^{\varkappa}}{h} \neq 0, \infty,$$

that is

$$\rho(x_1, x_2) = O^*(H^{1-\varkappa}) \quad , \quad H \to 0, \qquad 0 < \varkappa < 1.$$

It is obvious that (see (7))

$$\rho(x_1, x_2) \in C(\omega(P) \cap D \cup L) \cap C^1(\omega(P) \cap D),$$

because $0 < \varkappa < 1$.

Besides we have:

$$\rho(x_1, x_2) \frac{\partial h(x_1, x_2)}{\partial x_\alpha} = \rho \frac{\partial h}{\partial H} \frac{\partial H}{\partial x_\alpha} = \tag{8}$$

$$= O^*(H^{1-\varkappa})\, O^*(H^{\varkappa-1}) \left(\frac{\partial H}{\partial n}\right)^{-1} \frac{\partial H}{\partial x_\alpha} \quad , \quad H \to 0, \qquad \alpha = 1, 2$$

and

$$h(x_1, x_2) \frac{\partial \rho(x_1, x_2)}{\partial x_\alpha} = h \cdot h^{-1} \frac{\partial H}{\partial x_\alpha} = \frac{\partial H}{\partial x_\alpha}, \qquad \alpha = 1, 2. \tag{9}$$

From (8) and (9) it follows that $\rho(x_1, x_2)$, defined by (7), satisfies the condition 2).

Finally we have:

$$\left. h(x_1, x_2) \frac{\partial \rho(x_1, x_2)}{\partial n} \right|_{(x_1, x_2)=P} = \left. \frac{\partial H}{\partial n} \right|_P \neq 0$$

and the condition 3) is also fulfilled.

5. Examples

Let us now consider the prismatic shells whose thicknesses are given by the following equations:

$$2\,h(x_1\,,x_2)=h_0\,x_2^{\varkappa}\,,\qquad x_2>0 \tag{10}$$

$$2\,h(x_1\,,x_2)=h_0\,(1-r^2)^{\varkappa}\,,\qquad r<1 \tag{11}$$

$$2\,h(x_1\,,x_2)=h_0\,e^{-\varkappa/x_2}\,,\qquad x_2>0 \tag{12}$$

$$2\,h(x_1\,,x_2)=h_0\,e^{-\varkappa/(1-r^2)}\,,\qquad r<1 \tag{13}$$

where h_0 and $\varkappa$ are positive constants and

$$r^2=x_1^2+x_2^2\,.$$

In the cases (10), (12), the boundary L contains an open interval of the x_1-axis and in the cases (11), (13) the boundary contains an open arc of the circle $r=1$ or the whole circle.

It is obvious that in the cases (10), (11) the points of the above mentioned open interval or of the open arc, respectively, are blunt cusps for $\varkappa<1$ and sharpened cusps for $\varkappa\geq 1$. In the cases (12), (13) these points are always sharpened cusps.

In the cases (10)–(13) equation (2) has, respectively, the following forms:

$$h_0\,x_2^{\varkappa}\,\Delta v_3+\varkappa h_0\,x_2^{\varkappa-1}\,\frac{\partial v_3}{\partial x_2}+\frac{1}{\mu}\,\mathring{X}_3=0\,,\qquad (x_1\,,x_2)\in D\,, \tag{14}$$

$$h_0\,(1-r^2)^{\varkappa}\,\Delta v_3-2\,\varkappa h_0\,(1-r^2)^{\varkappa-1}\,x_\alpha\,\frac{\partial v_3}{\partial x_\alpha}+\frac{1}{\mu}\,\mathring{X}_3=0\,,\qquad (x_1\,,x_2)\in D\,, \tag{15}$$

$$h_0\,e^{-\varkappa/x_2}\,\Delta v_3+\varkappa h_0\,x_2^{-2}\,e^{-\varkappa/x_2}\,\frac{\partial v_3}{\partial x_2}+\frac{1}{\mu}\,\mathring{X}_3=0\,,\qquad (x_1\,,x_2)\in D\,, \tag{16}$$

$$h_0\,e^{-\varkappa/(1-r^2)}\,\Delta v_3-2\,\varkappa h_0\,x_\alpha\,(1-r^2)^{-2}\,e^{-\varkappa/(1-r^2)}\,\frac{\partial v_3}{\partial x_\alpha}+\frac{1}{\mu}\,\mathring{X}_3=0\,,\qquad (x_1\,,x_2)\in D\,. \tag{17}$$

In the cases (10), (11) we can assume $\rho(x_1\,,x_2)$, respectively, equal to [6]

$$h_0^{-1}\,x_2^{1-\varkappa}\,, \tag{18}$$

$$h_0^{-1}\,(1-r^2)^{1-\varkappa} \tag{19}$$

(6) See (7) and the remark at the end of the paragraph 3.

and after having multiplied equation (14) by the function (18) and equation (15) by the function (19), we obtain

$$(20) \qquad x_2 \Delta v_3 + \varkappa \frac{\partial v_3}{\partial x_2} + \frac{x_2^{1-\varkappa}}{\mu h_0} \mathring{X}_3 = 0 , \qquad (x_1 , x_2) \in D ,$$

$$(21) \qquad (1 - r^2) \Delta v_3 - 2 \varkappa x_\alpha \frac{\partial v_3}{\partial x_\alpha} + \frac{(1 - r^2)^{1-\varkappa}}{\mu h_0} \mathring{X}_3 = 0 , \qquad (x_1 , x_2) \in D .$$

System (1) may also be simplified in the same way.

Finally Fichera's function for the equations (20), (21), (16), (17) assumes, respectively, the forms:

$$b (x_1 , 0) = \varkappa - 1 \quad , \quad b (x_1 , x_2) |_{r=1} = 2 (\varkappa - 1) ,$$

$$b (x_1 , 0) = 0 \quad , \quad b (x_1 , x_2) |_{r=1} = 0 .$$

Thus, in a blunt cusp Fichera's function for the equations (20), (21). (16). (17) is negative and in a sharpened cusp is nonnegative.

6. REMARK

In a forthcoming work under some convenient assumptions the following statement will be proved.

The bounded displacement (i.e. the generalized solution in some sense of the system (1), (2) with boundary value data on a piece of boundary or on whole one where $h > 0$ or $b < 0$) of the shell is uniquely defined through its values at the regular points and at the blunt cusps of the boundary.

Thus, if the displacement will be given on the boundary, the sharpened cusps on the shell boundary will be free from the boundary conditions. This fact is in good agreement with physical intuition.

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