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<http://www.bdim.eu/item?id=RLINA_1980_8_68_5_407_0>

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**Geometria algebrica. — On k-dimensional elliptic scrolls** (*)

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**RIASSUNTO. —** Le varietà algebriche proiettive complesse, non singolari, di dimensione \( k \geq 3 \) (grado \( d > 3 \)), a curva sezione ellittica, o sono razionali o sono fasci ellittici di spazi lineari.

Le varietà del primo tipo sono state studiate e classificate da Enriques (cfr. [2], [3]) e Scorza (cfr. [8]); alle varietà del secondo tipo è dedicata la presente Nota.

Si illustrano alcune proprietà delle varietà fibrate in spazi lineari su di una curva ellittica, e si studiano i loro modelli linearmente normali \( W \). Indicati con \( d \) e \( k \) il grado e la dimensione di una siffatta \( W \) e con \( n \) la dimensione del minimo spazio di appartenenza, si dimostra che \( d \geq 2k + 1 \), \( d = n - 1 \). Infine, assegnata la curva ellittica base, si costruisce un modello esplicito di una \( W \) del tipo considerato per una qualunque dimensione \( k \) e per un qualunque grado \( d \geq 2k + 1 \).

1. — This paper is concerned with \( k \)-dimensional projective irreducible and complex algebraic varieties with elliptic curve sections with special regard to \( k \)-dimensional elliptic scrolls.

The \( k \)-dimensional varieties \( W \subset \mathbb{P}^n \) with elliptic curve sections were studied by Castelnuovo (cf. [1]) for \( k = 2 \), Enriques (cf. [2], [3]) for \( k = 3 \) and Scorza (cf. [8], [9]) for \( k \geq 3 \). Substantially they show that such varieties (but cones and cubic hypersurfaces) are either elliptic pencils of \( \mathbb{P}^{k-1} \)'s or rational varieties.

The rational case is deeply analyzed in the previously quoted works by Enriques and Scorza (see also [7], pp. 59-60).

The present paper is devoted to the study of the elliptic case (not developed in the classical works).

Specifically here we consider an irreducible smooth \( k \)-dimensional \( (k \geq 3) \) complex algebraic variety \( W \subset \mathbb{P}^n \) with elliptic curve sections. In sec. 3 we notice that if \( W \) is neither a rational variety nor a cubic hypersurface, then it is an elliptic \( k \)-scroll. In sec. 4 we show that the linearly normal models of such \( W \)'s are varieties of degree \( d \) in \( \mathbb{P}^{d-1} \) (non-hyperplane). We complete this result by proving (Theorem 4.2) that a \( k \)-dimensional \( (k \geq 2) \) linearly normal variety \( W_d \subset \mathbb{P}^n \) of degree \( d \) with elliptic curve sections is an elliptic

(*) Lavoro eseguito nell’ambito dell’attività del G.N.S.A.G.A. del C.N.R.
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(***) Nella seduta del 10 maggio 1980.
In sec. 5 we prove that an elliptic \( k \)-scroll \( (k \geq 2) \) \( W_d \subset \mathbb{P}^{d-1} \) has degree \( d \geq 2k + 1 \), and we show that for every admissible \( d \) there exists an elliptic \( k \)-scroll of degree \( d \), by constructing an explicit model of it.

2. From now on, the word variety will always mean irreducible complex projective algebraic variety. Such a variety of dimension \( k = 1 \) or \( k = 2 \) will be called curve or surface respectively.

Let \( V \) be a smooth variety of any dimension \( k \), and \( |D| \) the complete linear system associated to a divisor \( D \) on \( V \). We shall denote by \( \mathcal{O}_V \) the structural sheaf of \( V \); by \( \mathcal{O}_V(D) \) the invertible sheaf associated to \( D \); by \( h^q(\mathcal{O}_V(D)) = h^q(D) \) the complex dimension of the \( q \)-th cohomology vector space \( H^q(V, \mathcal{O}_V(D)) = H^q(\mathcal{O}_V(D)) \). In the case \( k \geq 2 \) consider an irreducible and smooth hypersurface \( S \) on \( V \); it is known that there exists a divisor \( D' \) on \( V \), linearly equivalent to \( D \) (briefly \( D' \equiv D \)) which does not contain \( S \) as a component and is transversal to \( S \). By \( D \cdot S \) we denote the divisor on \( S \) (defined mod linear equivalence) which \( D' \) cuts out on \( S \). For \( k = 2 \) the symbol \( (D \cdot C) \) represents the intersection index of the divisors \( D \) and \( C \) on the surface \( V \); if \( D \equiv C \), we also write \( (D - C) = (C^2) \).

Now let \( B \) be a smooth curve and \( \mathbb{P}^r \) the \( r \)-dimensional complex projective space. From now on, by saying that a variety \( V \) is contained in \( \mathbb{P}^r \) (or by writing \( V \subset \mathbb{P}^r \)) we mean that \( V \) is in \( \mathbb{P}^r \) but not in any hyperplane of its.

A \( k \)-dimensional scroll over \( B \) (briefly a \( k \)-scroll) is a smooth variety \( V \) embedded in some \( \mathbb{P}^r \) endowed with a morphism \( \pi: V \rightarrow B \) such that \( F_b = \pi^{-1}(b) \) is a \( \mathbb{P}^{k-1} \), for each \( b \in B \). If \( B \) is an elliptic curve, such a \( V \) is said an elliptic \( k \)-scroll.

Finally we introduce some other notations occurring in the sequel. On a \( k \)-dimensional smooth variety \( V \) consider a very ample divisor \( H \). Obviously it is always possible to choose \( i \ (i = 1, \ldots, k - 1) \) smooth irreducible hypersurfaces \( H_1, H_2, \ldots, H_i \), belonging to \( |H| \), pairwise transversal and such that the \((k - i)\)-dimensional algebraic characteristic cycle \( H_{k-i} = H_1 \cap \cdots \cap H_i \) (defined by intersecting these \( i \) hypersurfaces) is irreducible and smooth. We denote also by \( E \) the characteristic cycle \( H_1 \).

By means of well-known facts holding for complex smooth varieties, some standard cohomological calculations (see [4], sec. 2) give

**Remark 2.1.** If \( k \geq 3 \) and \( h^1(\mathcal{O}_{H_k}) = 0 \), then \( h^0(\mathcal{O}_V(H)) = h^0(\mathcal{O}_{H_2}(E)) + k - 2 \).

**Remark 2.2.** Suppose \( k \geq 3 \) and \( h^1(\mathcal{O}_{H_k}) = 1 \), \( h^2(\mathcal{O}_{H_k}) = h^1(\mathcal{O}_{H_2}(E)) = 0 \). Then \( h^0(\mathcal{O}_V(H)) = h^0(\mathcal{O}_{H_2}(E)) + s \), where \( 0 \leq s \leq k - 2 \).

(1) Remark that a \( k \)-scroll is a model of the projectivized \( \text{Proj}(\mathcal{E}) \) of a vector bundle \( \mathcal{E} \) of rank \( k \) over \( B \).
3. In [8] G. Scorza studied the $k$-dimensional varieties $W \subset \mathbf{P}^n$ with elliptic curve sections. If such a variety $W$ is not a cone and if $\deg W > 3$, then he shows that $W$ must be either a rational variety or an elliptic pencil of $\mathbf{P}^{k-1}$'s. Now, recalling the notations introduced in sec. 2 and denoting by $\Phi_H$ the closed immersion defined by $|H|$, this result can be restated as follows.

**Theorem 3.1. (Enriques-Scorza).** Let $V$ be a $k$-dimensional smooth variety $(k \geq 3)$ endowed with a very ample linear system $|H|$ the general characteristic 1-cycle $E$ of which is an elliptic curve. Then only the following cases can occur:

1. $V$ is isomorphic, via $\Phi_H$, to a smooth cubic hypersurface of $\mathbf{P}^{k+1}$;
2. $V$ is a rational variety;
3. $V$ is isomorphic, via $\Phi_H$, to an elliptic $k$-scroll over $E$.

Another proof, different from the original one, can be found in [4].

**Remark 3.1.** If $E$ is an elliptic curve, the 2-dimensional characteristic cycle $H^2$ is either a rational surface or it is isomorphic to an elliptic 2-scroll (Castelnuovo [1]; for a different proof working over any algebraically closed field see also [6], Theorem 3.1). From the proof of Theorem 3.1 in [4], one sees that the former case corresponds to $i)$ and $ii)$, whilst the latter case corresponds to $iii)$. In the former case Remark 2.1 shows that the linearly normal models $W_d = \Phi_H(V)$ of the $V$'s occurring in case $ii)$ are varieties of degree $d = (E^2)$ in $\mathbf{P}^{d+k-2}$.

For the sequel we need also the following

**Remark 3.2.** Let $W \subset \mathbf{P}^n$ be a $k$-scroll $(k \geq 3)$ over a (smooth) curve $B$ and let $H'$ be a hyperplane section of its. Then there happens: either $i)$ $H'$ is a $(k-1)$-scroll over $B$, or $ii)$ $H' = F_{b_1} + \cdots + F_{b_p} + S$ where $F_{b_i}$ is a fibre of $W$ and $S$ is a $(k-1)$-scroll over $B$ (2), (cf. [4]).

4. Theorem 3.1 is particularly meaningful when $H$ is a general hyperplane section of a smooth $k$-dimensional variety $W \subset \mathbf{P}^n$. In this case it says that a smooth variety $W_d \subset \mathbf{P}^n$ of degree $d > 3$ with elliptic curve sections is either a rational variety or an elliptic $k$-scroll. The rational case (case $ii)$ of Theorem 3.1) has been studied, for $k = 3$, by Enriques in [2] (see also [3]). He proved that the linearly normal models of the rational threefolds with elliptic curve sections are varieties $W_d \subset \mathbf{P}^{d+1}$ of degree $d$ ($4 \leq d \leq 8$) representable on $\mathbf{P}^3$ by a linear system of quadric or cubic surfaces. For $k > 3$, the linearly normal models in the rational case have been studied by Scorza (cf. [8]). They are, for $k = 4$, a $W_5 \subset \mathbf{P}^5$, and a $W_6 \subset \mathbf{P}^6$; for $k = 5$, a

(2) In a sense this Remark has a converse (see [4], Prop. 3.1); it could also be easily proven (cf. the following Theorem 5.1) that $1 \leq r \leq \deg W - 2k + 1$. 

$W_6 \subset \mathbb{P}^6$; for $k = 6$ the Grassmannian $W_6 \subset \mathbb{P}^6$, and, for each $k \geq 3$, there is always the complete intersection $W_4 \subset \mathbb{P}^{k-2}$ of two quadrics.

This sec. is meanly devoted to the analysis of elliptic $k$-scrolls, i.e. we are dealing with the case $iii$) of Theorem 3.1.

Now consider a $k$-dimensional smooth variety $V (k \geq 2)$, a complete very ample linear system $|H|$ on $V$, and its characteristic cycles $H_2$ and $H_1 = E$. From now on suppose $E$ is an elliptic curve; so we know (cf. [6]) $H_2$ is either a rational surface or an elliptic geometrically ruled surface (to wit it is isomorphic to an elliptic 2-scroll). From now on suppose also we are in the latter case; so, if $k \geq 3$ all we said in Remark 3.1 implies that the variety $W_d = \Phi_{H}(V)$ is a linearly normal elliptic $k$-scroll of degree $d = (E^2)$; note that this conclusion holds also for $k = 2$ (cf. [6]). It is known (cf. [6], p. 91) that

$$h^0(\mathcal{O}_{H_2}(E)) = d$$

and $h^1(\mathcal{O}_{H_2}(E)) = 0$; moreover, as $H_2$ is supposed to be an elliptic ruled surface if $k \geq 3$, we can apply Remark 2.2 and conclude that

$$h^0(\mathcal{O}_{V}(H)) = d + s,$$

where $0 \leq s \leq k - 2$. So, $a$ priori, $W_d$ is a linearly normal $k$-scroll of degree $d$ in $\mathbb{P}^{d+s-1}$. Really, for each $k \geq 2$ (if $k = 2$ see (4.1)), we can prove the following

**Theorem 4.1.** $W_d$ is contained in $\mathbb{P}^{d-1}$ (i.e. $s = 0$).

**Proof.** In case $k = 2$ our thesis follows immediately from all we said; so we prove the theorem by induction on $k$.

**Step 1.** There results $s \leq 1$. By Remark 3.2 the general hyperplane section of $W_d \subset \mathbb{P}^{d+s-1}$ is an elliptic $(k - 1)$-scroll $H_d \subset \mathbb{P}^{d+k-2}$ of degree $d$. Were $H_d$ linearly normal, the dimension of the corresponding embedding space should be $n = d - 1$, by induction; if not, it would be $n < d - 1$. So we must have $n = d + s - 2 \leq d - 1$.

**Step 2.** There results $s = 0$. By absurd suppose $s = 1$, i.e. $W_d \subset \mathbb{P}^d$. We are proving this fact can not occur by analyzing the following three cases:

$$(a)\quad d < 2k - 1; \quad (b)\quad d \geq 2k; \quad (c)\quad d = 2k - 1.$$ 

In case (a) the thesis is trivial; in fact two distinct fibres $F_1 = \mathbb{P}^{k-1}$ and $F_2 = \mathbb{P}^{k-1}$ of $W_d$ generate a $\mathbb{P}^{k-1}$.

In case (b) the linear span $\langle F_a, F_b \rangle$ of two fibres $F_a$ and $F_b$ of $W_d$ has codimension $\geq 1$; so there exists a hyperplane $\Pi \supset \langle F_a, F_b \rangle$. In view of Remark 3.2, we have

$$\Pi \cap W_d = F_a + F_b + F_{a_1} + \cdots + F_{a_t} + S,$$
where $F_{a_1}, \ldots, F_{a_i}$ are fibres of $W_d$ and $S$ is an elliptic $(k - 1)$-scroll of degree $d - 2 - i$. Suppose $S$ is contained in $\mathbb{P}^n$ (and not in any hyperplane of its). By induction

$$n \leq d - 2 - i - 1.$$ 

(4.3)

Denote by $L = \langle S, F_{a_1}, \ldots, F_{a_i}, F_a \rangle$ the linear span of $S$ and $F_{a_1}, \ldots, F_{a_i}, F_a$. Remark that $S$ cuts out on each fibre $F$ of $W_d$ the $\mathbb{P}^{k-2}$ which is the corresponding fibre of $S$; there follows that $\dim L \leq n + i + 1$; recalling (4.3), $\dim L \leq d - 2$. So there is a pencil $\{\Pi_t \in \mathbb{P}^1\}$ of hyperplanes $\Pi_t \subset \mathbb{P}^d$ through $L$, at least. Such a $\Pi_t$ intersects $W_d$ along

$$S + F_{a_1} + \cdots + F_{a_i} + F_a + R,$$

where $R$ is nothing but a fibre of $W_d$, by Remark 3.2. So it is defined a non-constant morphism from $\mathbb{P}^1$ to the elliptic base $B$ of $W_d$. This is absurd.

In case (c), in order to exclude the varieties $W_d \subset \mathbb{P}^d$ ($d = 2k - 1$) consider a hyperplane $\Pi$ containing a fibre $F$ of $W_d$ and the corresponding section $\Pi \cap W_d = F + S$; by Remark 3.2, $S$ is either a $(k - 1)$-scroll or a $(k - 1)$-scroll plus some fibres $F_{a_1}, \ldots, F_{a_i}$. The latter case does not occur: otherwise the fibres $F_{a_1}$ and $F$, contained in $\Pi = \mathbb{P}^{k-2}$, should meet. Suppose $S$ is contained in $\mathbb{P}^n$ (and not in any hyperplane of its). As $S$ has degree $d - 1$, by induction we have $n \leq d - 2$ (3); hence there exists a pencil $\{\Pi_t \in \mathbb{P}^1\}$ of hyperplanes $\Pi_t \subset \mathbb{P}^d$ through the $\mathbb{P}^n$ containing $S$. Such a $\Pi_t$ intersects $W_d$ along $S + R$, where, by Remark 3.2, $R$ is nothing but a fibre of $W_d$. Thus we can conclude, as in case (b).

There follows immediately

**Remark 4.1.** Let $W$ be an elliptic $k$-scroll and $H'$ a general hyperplane section of its. Then $H'$ and each other general characteristic cycle of $|H'|$ is not linearly normal.

Let $W_d \subset \mathbb{P}^n$ be a smooth $k$-dimensional linearly normal variety ($k \geq 2$) of degree $d$ with elliptic curve sections. If $W_d$ is a $k$-scroll, Theorem 4.1 shows that $n = d - 1$. On the other side if $n = d - 1$ and $k \geq 3$, $W_d$ can be neither a cubic hypersurface nor a rational variety (see Remark 3.1). Hence, by Theorem 3.1, $W_d$ is an elliptic $k$-scroll. The same conclusion holds if $k = 2$ (cf. [6] Corollary 3.1). So we can state

**Theorem 4.2.** Let $W_d \subset \mathbb{P}^n$ be a $k$-dimensional linearly normal smooth variety ($k \geq 2$) of degree $d$ with elliptic curve sections. Then $W_d$ is an elliptic $k$-scroll if and only if $n = d - 1$.

5. - As we already said the number of possible varieties occurring in case ii) of Theorem 3.1 decreases as the dimension $k$ of the variety increases. On the contrary here we show that the degree $d$ of the varieties possibly

(3) Really it is $n = d - 2$; otherwise two fibres of $S$ should meet.
occurring in case \( iii \) of Theorem 3.1 must satisfy the bare condition \( d \geq 2k + 1 \) and that for every admissible \( d \) there exists an elliptic \( k \)-scroll of degree \( d \). This latter fact will be proved by exhibiting an explicit projective model. We start with the following

**Proposition 5.1.** Let \( W_d \subset \mathbb{P}^{d-1} \) be an elliptic \( k \)-scroll \( (k \geq 2) \) of degree \( d \), then \( d \geq 2k + 1 \).

**Proof.** This fact is trivial for \( k = 2 \). Let us continue the proof by induction on \( k \). By absurd, let \( W_d \subset \mathbb{P}^{d-1} \) be an elliptic \( k \)-scroll of degree

\[
(5.1) \quad d < 2k + 1.
\]

Consider a fibre \( F \) of \( W_d \), a hyperplane \( \Pi \) containing \( F \) and the corresponding section

\[
(5.2) \quad \Pi \cap W_d = F + S.
\]

By reasoning as in the proof of Theorem 4.1 (Step 2, case (c)) we conclude that \( S \) is an elliptic \((k-1)\)-scroll of degree \( d-1 \). Consider the linearly normal \((k-1)\)-scroll \( S' \) of degree \( d-1 \) embedded by the complete linear system of the hyperplane sections of \( S \). By Theorem 4.2, \( S' \) is contained in \( \mathbb{P}^{d-2} \); so, by induction, \( d-1 \geq 2(k-1)+1 \). Recalling (5.1) it can only be \( d = 2k \). In this case consider the \((k-1)\)-scroll \( S \) in (5.2) and let \( L \) be a \( \mathbb{P}^{d-k} = \mathbb{P}^k \) containing the fibre \( F \). By applying repeatedly Remark 3.2 we see that \( L \cap S \) contains an elliptic curve \( C \), isomorphic to \( B \), which must be a section of \( W_d \) (in the sense that \( C \) intersects each fibre in one point only). But as \( F \) is a hyperplane in \( L \), \( \text{Card} (C \cap F) = \deg C \), and \( \deg C > 1 \), \( C \) being elliptic.

The bound given by Proposition 5.1 is as best as possible. In fact, for each \( k \geq 2 \), we are going to construct an elliptic \( k \)-scroll of degree \( d = 2k + 1 \). This construction generalizes the one given in [5] for the quintic elliptic \( 2 \)-scroll in \( \mathbb{P}^4 \).

In \( \mathbb{P}^{2k} \) consider \( k \) distinct 2-planes \( \pi_i \) \( (i = 1, \cdots, k) \) pairwise intersecting in a single common point \( p_0 \) and spanning the whole \( \mathbb{P}^{2k} \). Consider also an elliptic curve \( B \) and \( k \) distinct points \( b_1, \cdots, b_k \) of \( B \) such that each \( b_i - b_j \) \((i, j = 1, \cdots, k; i \neq j)\) is not of order three, i.e.

\[
(5.3) \quad 3(b_i - b_j) \neq 0 \quad (i, j = 1, \cdots, k; i \neq j).
\]

Let \( \eta_i : B \to \pi_i \) \( (i = 1, \cdots, k) \) be a closed immersion such that \( \eta_i(b_i) = p_0 \), \((i = 1, \cdots, k)\), and denote by \( B_i \) the elliptic cubic curve \( \eta_i(B) \). For each \( b \in B \), \( b \neq b_i \) consider the \( \mathbb{P}^{k-1} \)

\[
(5.4) \quad F_b = \langle \eta_1(b), \eta_2(b + b_2 - b_1), \cdots, \eta_k(b + b_k - b_1) \rangle
\]
spanned by the independent points \( \eta_i (b + b_i - b_1) \in B_i \ (i = 1, \cdots, k) \). Consider also the map

\[ \Phi : B \setminus \{b_i\} \to \text{Grass} (k - 1, 2k) \]

which takes values in the Grassmann manifold of the \( \mathbb{P}^{k-1} \)'s in \( \mathbb{P}^{2k} \), defined by \( \Phi (b) = F_b \). Denote by \( \Phi \) its extension to \( B \) and put

\[
(5.5) \quad F_{b_i} = \Phi (b_i).
\]

**Lemma 5.1.** If \( b, b' \in B \) are two distinct points, then \( F_b \cap F_{b'} = \emptyset \).

**Proof.** i) Suppose \( b, b' \in B \setminus \{b_i\} \). Put \( p_i = \eta_i (b + b_i - b_1) \) and \( p'_i = \eta_i (b' + b_i - b_1), \ (i = 1, \cdots, k) \). There results

\[ F_b = \langle p_1, \cdots, p_k \rangle \quad \text{and} \quad F_{b'} = \langle p'_1, \cdots, p'_k \rangle. \]

By absurd, suppose \( F_b \cap F_{b'} \neq \emptyset \); there follows

\[
(5.6) \quad \dim \langle F_b, F_{b'} \rangle \leq 2k - 2.
\]

Let \( l_i \) be the line \( \langle p_i, p'_i \rangle \) and suppose such lines are pairwise skew. The linear span \( S_{1,2} = \langle l_1, l_2 \rangle \) has dimension 3. Moreover \( S_{1,2} \subset \langle \pi_1, \pi_2 \rangle \); so \( S_{1,2} \cap l_3 \), which is contained in \( \langle \pi_1, \pi_2 \rangle \cap \pi_3 = \{ p_0 \} \), is either empty or reduced to \( p_0 \). But the latter case cannot happen and so

\[
(5.7) \quad S_{1,2} \cap l_3 = \emptyset.
\]

In fact, if \( S_{1,2} \cap l_3 = \{ p_0 \} \), then there were two lines \( l_3 \) and \( l_4 \) (or \( l_5 \)) through \( p_0 \), contradicting our assumption; to see this it is sufficient to prove that if \( S_{1,2} \ncontaining \( \pi_1 \cup \pi_2 \). So it cuts out on one of them (4), suppose \( \pi_1 \), the line \( l_4 \). But \( l_4 \) intersects the cubic curve \( B_i \) in \( p_1, p'_i \) and in a further point which must be \( p_0 \), as \( p_0 \in S_{1,2} \); hence \( p_0 \in l_4 \). Thus \( (5.7) \) is true and then \( S_{1,2,3} = \langle S_{1,2}, l_3 \rangle = \langle l_1, l_2, l_3 \rangle \) has dimension 5. By repeating the same argument we conclude

\[
(5.8) \quad \dim \langle l_1, l_2, \cdots, l_k \rangle = 2k - 1.
\]

As \( \langle F_b, F_{b'} \rangle = \langle l_1, \cdots, l_k \rangle \), (5.8) contradicts (5.6). Thus the lines \( l_i \) cannot be pairwise skew. Suppose \( l_i \cap l_j \neq \emptyset \); it must be \( l_i \cap l_j = \{ p_0 \} \), and so the triple \( p_0, p_i, p'_i \) (respectively \( p_0, p_j, p'_j \)) is collinear on \( B_i \) (respectively on \( B_j \)). This means that the pair \( b + b_i - b_1, b' + b_i - b_1 \) must belong to the

\( (4) \) It could be seen (cf. the following part of this proof) that if \( p_0 \in S_{1,2} \) and \( S_{1,2} \ncontaining \( \pi_1 \), then \( S_{1,2} \ncontaining \( \pi_2 \).
$g^i_2$ defined by $b_i$ on $B$ and that the pair $b + b_j - b_i, b' + b_j - b_i$ must belong to the $g^i_1$ defined by $b_j$ on $B$. Therefore it must be

$$b + b_i - b_i + b' + b_i - b_i + b_i = 0, b + b_j - b_i + b' + b_j - b_i + b_j = 0.$$  

But such relations cannot hold together by the assumption (5.3).

ii) Suppose $b' = b_i$. Assuming, by absurd, $F_b \cap F_{b'} \neq \emptyset$, reasoning similarly to the case i) and taking into account (5.5), one sees that the relations

$$b + b_i - b_i + 2 b_i = 0, b + b_j - b_i + 2 b_j = 0$$

must hold together for some $i$ and $j$ $(i \neq j)$. Again this contradicts (5.3).

Now put

$$(5.9) \quad W = \bigcup_{b \in B} F_b;$$

there holds the following

**PROPOSITION 5.2.** $W$ is an elliptic $k$-scroll over $B$ embedded in $P^{2k}$ and of degree $d = 2k + 1$.

*Proof.* By construction and Lemma 5.1 it follows immediately that (5.9) defines an elliptic $k$-scroll (over $B$) $W \subset P^{2k}$; we prove by induction that it has degree $d = 2k + 1$. For $k = 2$, $W$ is the quintic elliptic 2-scroll in $P^4$ (cf. [5], Prop. 5.1). Otherwise consider a hyperplane $H \subset P^{2k}$ containing the planes $\pi_1, \ldots, \pi_{k-1}$. The hyperplane $H$ cuts out on the cubic curve $B_k$ the point $p_0$ and two other points $p_1$ and $p_2$. Hence $W \cap H$ is constituted by the two fibres of $W$ through $p_1$ and $p_2$ and by the $(k-1)$-scroll $S$ generated by the cubic curves $B_1, \ldots, B_{k-1}$ (in the same way as $W$ is generated). So $\text{deg } W = 2 + \text{deg } S$, and by induction on $S$ we conclude.

The previous construction can be generalized and gives a model of an elliptic $k$-scroll of degree $d$ for each $d \geq 2k + 1$. Replace the 2-planes $\pi_i$ ($i = 1, \ldots, k$) by linear spaces $L_i$ of dimensions $r_i (r_i \geq 2)$ contained in $P^r$ ($r = \sum r_i$), pairwise intersecting in a single common point $p_0$ and generating the whole $P^r$. Replace the cubic curves $B_i$ by elliptic curves of degree $r_i + 1$ isomorphic to $B$ via closed immersions $\gamma_i$. Define the $(k-1)$-linear space $F_b$ as in (5.4). Lemma 5.1 continues to hold and so a formula analogous to (5.9) defines a variety $W$ which is easily seen to be an elliptic $k$-scroll over $B$ of degree $d = \sum_{i=1}^k r_i + 1 = r + 1$ contained in $P^r$. 

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