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Arturo V. Ferreira, Pier Daniele Napolitani

On invertible holomorphic functions with values in a topological algebra

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On invertible holomorphic functions with values in a topological algebra.* Nota di ARTURO V. FERREIRA e PIER DANIELE NAPOLITANI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studia l'equazione $e^F = f$ con f funzione olomorfa a valori in un algebra topologica non necessariamente commutativa; si costruisce un fascio $\mathscr{E}_f$ che dipende dalla f data nel caso non commutativo, e che permette di collegare l'esistenza di soluzioni all'annullamento del primo gruppo di coomologia a valori in $\mathscr{E}_f$.

1. Classical analytic function theory has been extended to more and more broad domains. However, even in the subject of vector-valued functions defined in finite-dimensional domains, there are several natural problems which at present have no satisfactory solution and deserve deeper analysis. Here we deal with such a problem; we shall be concerned with the equation $e^F = f$ where F, f stand for holomorphic functions defined on a complex analytic finite-dimensional manifold X, denumerable at infinity, with values in a complex unitary complete topological algebra A whose topology can be defined by a system of algebra seminorms. As a matter of fact the results below apply, and the proofs are also valid with slight modifications if X is a complex analytic space; in every way the more general statements thus obtained are already implied by the correspondent ones for manifolds by blowing up the singularities of X.

(*) Nella seduta del 10 maggio 1980.

That being, to be specific let $f: X \to A$ be a given holomorphic function which takes its values in the group $A^\star$ of invertible elements of A. We look for the existence of a holomorphic function $F: X \to A$ such that $\exp(F(z)) = f(z)$ for every z in X, where $\exp: A \to A^\star$ denotes the mapping which sends $a \in A$ into $e^{2\pi i a} \in A^\star$. In the sequel X may (and shall) be supposed connected without loss of generality because the equation under consideration is solved globally at once if we know a solution on every component of X.

If A reduces to the complex number field **C**, it is an elementary classical result that the obstructions to the resolution of our problem are computed by means of the first Čech cohomology group $H^1(X, \mathbf{Z})$. Hence, our problem can surely be solved if $H^1(X, \mathbf{Z}) = 0$, and the difference of two solutions F_1, F_2 is a constant function $\in \mathbf{Z}$ because X is supposed connected. All this information is contained in the exact sequence of cohomology groups:

$$0 \to \mathbf{Z} \to \mathcal{O}(X; \mathbf{C}) \xrightarrow{\exp} \mathcal{O}(X; \mathbf{C})^\star \to H^1(X, \mathbf{Z}) \to H^1(X, \mathcal{O}_X)$$

which results from the exact sequence of sheaves:

$$0 \to \mathbf{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\star \to 0$$

where $\mathcal{O}(X; \mathbf{C})$ (resp. $\mathcal{O}(X; \mathbf{C})^\star$) is the algebra of holomorphic functions on X (resp. which are invertible) and $\mathcal{O}_X$ (resp. $\mathcal{O}_X^\star$) the sheaf of holomorphic (resp. invertible holomorphic, functions on X; exp denotes the exponential mapping composed with the multiplication by $2\pi i$. In particular $H^1(X, \mathbf{Z}) \simeq \mathcal{O}(X; \mathbf{C})^\star/\exp(\mathcal{O}(X; \mathbf{C}))$ holds if $H^1(X, \mathcal{O}_X) = 0$, which is always the case when X is Stein.

The situation modifies somewhat if $A \neq \mathbf{C}$ is arbitrary commutative, and, as we shall see changes rather radically for A non-commutative. In the sequel notation and results of the paper "On first Čech groups H^0, H^1 of maximal ideal spaces" by E. Ballico, A. V. Ferreira and P. D. Napolitani (cf. this Rendiconti vol. LXVIII) will be used.

First denote by $\mathcal{O}(X; A)$ the algebra of holomorphic functions with values in A and by $\mathcal{O}(X; A)^\star$ the group of invertible elements of $\mathcal{O}(X; A)$. $\mathcal{O}_X(A)$ will be the sheaf of germs of A-valued holomorphic functions on X and $\mathcal{O}_X(A)^\star$ the subsheaf constituted by the invertible germs; we note that $\mathcal{O}_X(A)$ can be expressed as a completed topological tensor product $\mathcal{O}_X \hat{\otimes} A$ of the Fréchet nuclear sheaf $\mathcal{O}_X$ with the constant sheaf A. We consider first the case when A is commutative. In general, the kernel E of exp is no longer **Z** and U is $\neq A^\star$. However, the fact that E is a discrete subset of A will enable us to discuss the the situation along the same lines as in the scalar case $A = \mathbf{C}$.

We have the following easy result

LEMMA 1: *Suppose* Ω *is a connected coordinate domain in* $X, z_0 \in \Omega$, $f(z_0) \in U$ *and* $H^1(\Omega, \mathbf{Z}) = 0$. *Then, denoting by* F *the primitive of the exact vector-valued form* $f^{-1}\partial f$ *in* Ω *which takes in* z_0 *a value* a_0 *for which*

$\exp((1/2\pi i)\, a_0) = f(z_0)$, F *is a holomorphic function and* $\exp((1/2\pi i)\, F) = f$ *on* Ω.

Thus, denoting by $\mathcal{O}_X(U) \subset \mathcal{O}_X(A)$ the sheaf of germs of holomorphic functions taking values in U, we have an exact sequence of sheaves

$$0 \to E \to \mathcal{O}_X(A) \xrightarrow{\exp} \mathcal{O}_X(U) \to 0$$

from which we deduce the cohomology exact sequence

$$(1) \quad 0 \to E \to \mathcal{O}(X;A) \xrightarrow{\exp} \mathcal{O}(X;U) \to H^1(X, E) \to H^1(X, \mathcal{O}_X(A))$$

By taking into account that E is a torsion-free **Z**-module and $\mathcal{O}_X(A)$ is the completed tensor product of A by a nuclear Fréchet sheaf, from the preceeding exact sequence we obtain

THEOREM 1. *If* $H^1(X, \mathbf{Z}) = 0$ *the exponential map in* (1) *is surjective. If* $H^1(X, \mathcal{O}_X)$ *vanishes, which is always the case when* X *is Stein we have* $H^0(\Sigma(A), \mathbf{Z}) \otimes H^1(X, \mathbf{Z}) \simeq \mathcal{O}(X;U)/\exp(\mathcal{O}(X;A))$. *Henceforth let us suppose* $H^1(X, \mathcal{O}_X) = 0$. *Then, if* $A^\star$ *is open in* A, *the hypothesis* $H^1(\Sigma(A), \mathbf{Z}) = 0$ *implies* $H^0(\Sigma(A), \mathbf{Z}) \otimes H^1(X, \mathbf{Z}) \simeq \mathcal{O}(X;A)^\star/\exp(\mathcal{O}(X;A))$; *in the general case, if the projective limit of cohomology groups of partial spectra* $\varprojlim H^1(\Sigma(A_p), \mathbf{Z})$ *vanishes and* U *is closed in* $A^\star$, *or* $\mathrm{Im}(H^1(\Sigma(A)^\lambda, \mathbf{Z}) \to H^1(\Sigma(A)_\lambda, \mathbf{Z})) = 0$ *we have also that* $H^0(\Sigma(A), \mathbf{Z}) \otimes H^1(X, \mathbf{Z}) \simeq \mathcal{O}(X;A)^\star/\exp(\mathcal{O}(X;A))$.

Remark. If $A^\star$ is open in A and $H^1(X, \mathbf{Z}) = 0$, it is easy to see moreover that we have an exact sequence

$$H^1(\Sigma(A), \mathbf{Z}) \to H^1(X, \mathcal{O}_X(U)) \to H^1(X, \mathcal{O}_X(A)^\star) \to 0 .$$

2. As we have seen, when A is commutative the various obstructions for solving $\exp(F) = f$ come out essentially from the topology of the domain X and of the structure space of the algebra $\Sigma(A)$. In the non commutative case, the properties of the multiplicative structure of A must be taken into consideration.

The main difficulty arises from the fact that, even in the Banach case, the composition structure on small neighbourhoods of 1 that springs out from Campbell–Hausdorff formula is not suitable to express cocycles when we try to patch together local solutions of our exponential equation.

Thus, to continue the study of the equation $\exp(F) = f$ in the setting of known cohomology theories, some additional hypothesis are necessary. In particular, to apply usual sheaf cohomology we must restrain ourselves to consider only those holomorphic invertible functions f such that for any z in X, $f(z)$ is of the form $f(z) = e^x$ for some x in the closed full subalgebra $C_{f(z)}$ generated by $f(z)$.

Then we will consider the sheaf $\mathscr{F}_f$ (resp. $\mathscr{U}_f$) on X of germs of holomorphic functions $F(z)$ with $F(z) \in C_{f(z)}$ (resp. $F(z) \in \exp(C_{f(z)})$) and the sequence of sheaves

$$(\star) \qquad 0 \to \ker \exp \to \mathscr{F}_f \xrightarrow{\exp} \mathscr{U}_f \to 0$$

whose exactness will be a consequence of Lemma 3 below.

Let us denote by $\mathscr{E}_f$ the sheaf ker exp; $\mathscr{E}_f$ need not to be constant, nevertheless it possesses the following structural property:

LEMMA 2. *Let e be a section of $\mathscr{E}_f$ on a open set $\Omega \subset X$. Then, there exists a sequence of idempotent orthogonal sections of $\mathscr{E}_f$ over Ω, $(j_n)_{n \in \mathbf{Z}}$ such that*

$$1 = \sum_{n \in \mathbf{Z}} j_n \quad , \quad e = \sum_{n \in \mathbf{Z}} n j_n$$

in the normal convergence.

Proof. Take a point z in Ω; there will be a uniquely determined sequence of orthogonal idempotents with sum 1 in $C_{f(z)}$, $(j_n(z))_{n \in \mathbf{Z}}$ for which $e(z) = \sum_{n \in \mathbf{Z}} n j_n(z)$. The fact that for fixed $n \in \mathbf{Z}$ the function $j_n(z)$ is holomorphic in Ω and the normal convergence of the series results from the integral formula which expresses the dependence of $j_n(z)$ on $e(z)$ in each algebra $C_{f(z)}$.

LEMMA 3. *Let Ω be a simply connected domain in X and $a : \Omega \to A$ a holomorphic function such that $a(z) \in \exp(C_{f(z)})$ for every z in Ω. Then, there exists a holomorphic function $x : \Omega \to A$ which satisfies the equation $\exp(x) = a$ and $x(z) \in C_{f(z)}$ for $z \in \Omega$.*

Proof. First fix z_0 in Ω and consider also a fixed solution $x_0 = x(z_0)$ in $C_{f(z_0)}$ for the exponential equation. Also note that for y in A, we have $C_y = \varprojlim C_{\pi_p(y)}$ where $C_{\pi_p(y)}$ denotes for every $p \in \mathscr{N}$ the closed full subalgebra of A_p generated by $\pi_p(y)$.

Let now fix p in $\mathscr{N}$ and consider the datum at z_0, $\pi_p(x_0)$ and the equation $\exp(x(z)) = \pi_p(a(z))$. We know that point by point it has a solution x in the Banach algebra A_p such that $x(z) \in C_{\pi_p(f(z))}$. Then by an accurate use of the analytic calculus we deduce that in some neighbourhood V of z_0 a solution $x(z)$ can be constructed which depends holomorphically on z and verifies $x(z) \in C_{\pi_p(f(z))}$.

Also, it is clear that such a solution can be holomorphically extended out of V along every path in Ω. By considering all such holomorphic continuations and applying the monodromy theorem we conclude there is just one holomorphic function x_p globally defined on Ω which satisfies the relation $x_p(z) \in C_{\pi_p(f(z))}$ and has on V the determination produced by a definite schema of analytic calculus.

That being, it is not difficult to recognize that for each $z \in \Omega$ the family $(x_p(z))_{p \in \mathcal{N}}$ defines an element of A, $x(z)$ in $C_{f(z)}$ and $x(z)$ is a holomorphic function of z.

Thus, we can state

THEOREM 2. *We have the cohomology exact sequence*

$$0 \to E_f \to C_f \xrightarrow{\exp} U_f \to H^1(X, \mathscr{E}_f) \to H^1(X, \mathscr{F}_f)$$

where C_f *(resp.* $E_f : U_f$*) stands for the space of* A*-valued holomorphic functions whose value at each point* z *of* X *belongs to* $C_{f(z)}$ *(resp. is a linear combination with integer coefficients of idempotents in* $C_{f(z)}$*; the exponential of some element in* $C_{f(z)}$*). In particular, when* $H^1(X, \mathscr{E}_f) = 0$*, the equation* $e^F = f$ *has a solution* $F \in \mathcal{O}(X; A)$.

Remark. It should be observed that in a extreme variety of important instances the condition $H^1(X, \mathbf{Z}) = 0$ ensure already $H^1(X, \mathscr{E}_f) = 0$ for any $f \in \mathcal{O}(X; A)^\star$, or any f such that $f(z) \in U$, $z \in X$. This is mainly the case for most algebras of spectral operators; the case of a matrix algebra is covered by our remark so that some known results (such as Cartan's lemma on holomorphic matrices) are automatic corollaries to the above theorem.

Let us point out furthermore that the sheaf $\mathscr{E}_f$ reduces to a constant sheaf E without torsion in the following algebraic situations : (*i*) The set I of idempotents of A is a discrete topological subspace of A (in particular, if I is contained in the center of A); (*ii*) A does not contain non-trivial (algebraically) nilpotent elements; (*iii*) idempotents and (algebraically) nilpotent elements do commute.