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On the ϕ-Stability for differential systems

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Equazioni differenziali ordinarie. — *On the φ-Stability for differential systems.* Nota di OLUSOLA AKINYELE, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — L'Autore dà le definizioni di φ–stabilità per sistemi di equazioni differenziali e dà condizioni sufficienti perché queste stabilità abbiano luogo.

§ 1. INTRODUCTION

The most important technique to date in the theory of non-linear differential equations is the second method of Lyapunov. The method has been widely used to study differential systems of various kinds including functional differential equations and differential equations in abstract spaces. Furthermore, the power of the applications of differential inequalities together with Lyapunov's second method was demonstrated in the book of Lakshminkantham and Leela [7]. Several other authors [2, 4, 5, 6, 8, 9] have validated the possibility of applying this method including their modifications and generalizations for various problems of stability and boundedness of diffential systems. Sometimes [4, 5] whenever a new type of stability was introduced, a modification or generalization of the known basic comparison result was required to study such stability concepts.

In this work, in § 2 we introduce new definitions of φ-stability for differential systems which include definitions of the Lyapunov stability and several other known generalization of Lyapunov stability. To study these new stability concepts we develop also in § 2 a new comparison principle which is of a very general nature and includes as special cases well-known comparison results and their generalizations. In § 3, employing our comparison theorem and differential inequalities we finally give sufficient conditions for our concepts of stability to hold. Our results contain as special cases some well-known results of [1, 3, 7].

§ 2. BASIC DEFINITIONS AND NOTATIONS

We shall consider the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x) \qquad x(t_0) = x_0 \tag{1}$$

where $f \in C(R^+ \times R^n, R^n)$. Here $R^+ = [0, \infty)$, R^n denotes the Euclidean n-space and $C(R^+ \times R^n, R^n)$ the class of continuous functions from $R^+ \times R^n$

(*) Nella seduta del 12 aprile 1980.

to R^n. For any $\rho > 0$ let $S_\rho = \{x \in R^n : \|x\| < \rho\}$, $\|\cdot\|$ being any norm on R^n. In what follows we assume conditions on f which guarantee the existence and uniqueness of solutions of (1) and that $f(t, 0) = 0$.

DEFINITION 2.1. A function α belongs to class $\mathscr{K}$ if $\alpha \in C([0, \rho), R^+)$ such that $\alpha(r)$ is strictly monotone increasing in r and $\alpha(0) = 0$.

DEFINITION 2.2. A function $\eta(t, r)$ belongs to the class $\mathscr{K} \times \mathscr{K}$ if $\eta \in C(R^+ \times [0, \rho), R^+)$, $\eta \in \mathscr{K}$ for each $t \in R^+$ and η is monotone increasing in t for each $r > 0$ and $\eta(t, r) \to \infty$ as $t \to \infty$ for each $r > 0$.

DEFINITION 2.3. A function φ is said to belong to class $\mathscr{D}$ if φ is a continuous function defined on $R^+ \times R^n$ into R^n which is monotonically increasing and partially differentiable in the variable t on R^+ for each $x \in R^n$ such that $\varphi(t, \cdot) \geq 1$ for $0 \leq t < \infty$, $\lim_{t \to \infty} \varphi(t, \cdot)$ exists and $\lim_{t \to \infty} \varphi(t, \cdot) = C \geq 1$ where C is a real number.

DEFINITION 2.4. The trivial solution $x = 0$ of the system (1) is said to be

φS_1: φ-equistable if given any $\varepsilon > 0$, $t_0 \in R^+$ there exist a positive function $\delta = \delta(t_0, \varepsilon)$ that is continuous in t_0 for each ε, a positive number A and $\varphi \in \mathscr{D}$ such that for any solution $x(t, t_0, x_0)$ of the system (1)

$$\|\varphi(t_0, x_0)\| \leq \delta$$

implies

$$\|\varphi(t, x(t, t_0, x_0))\| < A\varepsilon, \qquad t \geq t_0;$$

φS_2: φ-uniformly stable if the δ in φS_1 is independent of t_0,

φS_3: φ-quasi-equi asymptotically stable if for each $\varepsilon > 0$, $t_0 \in R^+$ there exist positive numbers $\delta_0 = \delta_0(t_0)$, $A > 0$, $T = T(t_0, \varepsilon)$ and $\varphi \in \mathscr{D}$ such that for $t \geq t_0 + T$,

$$\|\varphi(t_0, x_0)\| \leq \delta_0$$

implies

$$\|\varphi(t, x(t, t_0, x_0))\| < A\varepsilon,$$

φS_4: φ-quasi uniformly asymptotically stable if δ_0 and T in φS_3 are independent of t_0,

φS_5: φ-equi asymptotically stable if φS_1 and φS_3 hold together;

φS_6: φ-uniformly asymptotically stable if φS_2 and φS_4 hold together.

Remark 1. If for $\varphi \in \mathscr{D}$, $\varphi(t, x(t, t_0, x_0) \equiv x(t, t_0, x_0)$, then φS_1—φS_6 become Lyapunov stability, [cf. 7].

Denote by $\mathscr{G}$ the class of functions $h \in C(R^+, R^+)$ such that $h(t)$ is monotonically increasing and differentiable in R^+ such that $h(t) \geq 1$ for $t \in R^+$ and $\lim_{t \to \infty} h(t) = b \geq 1$, b being a real number. If for $\varphi \in \mathscr{D}$ and $k \geq 0$,

$\varphi(t, x(t, t_0, x_0)) = h^k(t)\, x(t, t_0, x_0)$ then definitions φS_1—φS_6 reduce to stability with respect to h of degree k [3]. If $\varphi(t, x(t, t_0, x_0)) = x^n(t, t_0, x_0)$ $n = 0, 1, 2, \cdots, p$ where $x^{(n)}(t, t_0, x_0)$ denotes the n^{th} derivative of $x(t, t_0, x_0)$ then φS_1—φS_6 reduce to stability of order p [3]. In general if $\varphi(t, x(t)) = h^k(t)\, x^{(n)}(t, t_0, x_0)$ for $k \geq 0$, $h \in \mathcal{G}$ and $n = 0, 1, 2, \cdots, p$, then φS_1—φS_6 reduce to stability with respect to the function h of degree k and order p [3], where (1) has p-times differentiable solutions.

Definition 2.4 therefore generalizes many stability concepts including Lyapunov stability.

THEOREM 2.5. (Generalized comparison theorem). *Assume that*

(i) $V \in C(R^+ \times R^n, R^+)$, $V(t, x)$ *is locally Lipschitzian in* x *for each* $t \in R^+$,

(ii) $g \in C(R^+ \times R^+, R)$, $g(t, u)$ *is nondecreasing in* u *for each* $t \in R^+$ *and the maximal solution* $r(t, t_0, u_0)$ *of the scalar differential equation*

$$\frac{du}{dt} = g(t, u) \qquad u(t_0) = u_0 \geq 0 \tag{2}$$

exists to the right of t_0,

(iii) *for any solution* $x(t, t_0, x_0)$ *of* (1) *there exists* $\varphi \in \mathcal{D}$ *which is locally Lipschitzian in* x *for each* $t \in R^+$ *and for* $(t, \varphi(t, x(t))) \in R^+ \times R^n$,

$$\begin{aligned} D^+ V(t, \varphi(t, x)) = \limsup_{h \to 0} \frac{1}{h} \Big[V\Big(t + h, \varphi(t, x(t) + hf(t, x)) + \\ + h \frac{\partial \varphi}{\partial t}(t, x(t) + hf(t, x))\Big) - V(t, \varphi(t, x(t)))\Big] \leq \\ \leq g(t, V(t, \varphi(t, x(t)))) . \end{aligned} \tag{3}$$

Then if $x(t, t_0, x_0)$ is any solution of (1) existing for $t \geq t_0$, such that $V(t_0, \varphi(t_0, x_0)) \leq u_0$, then

$$V(t, \varphi(t, x(t, t_0, x_0))) \leq r(t, t_0, u_0) \qquad \text{for} \quad t \geq t_0 .$$

Proof. For $t \geq t_0$ and $\varphi \in \mathcal{D}$ define

$$m(t) = V(t, \varphi(t, x(t, t_0, x_0)))$$

then, for $h \geq 0$ sufficiently small,

$$\begin{aligned} m(t+h) - m(t) \leq L(t+h) \,\| \varphi(t+h, x(t+h)) - \varphi(t+h, x(t) + hf(t, x)) \| \\ + L(t+h) \,\| \varphi(t+h, x(t) + hf(t, x)) - \varphi(t, x(t) + hf(t, x)) \\ - h \frac{\partial \varphi}{\partial t}(t, x(t) + hf(t, x)) \| \\ + V(t+h, \varphi(t, x(t) + hf(t, x)) + h \frac{\partial \varphi}{\partial t}(t, x(t) + hf(t, x))) - V(t, \varphi(t, x(t))) \end{aligned}$$

where L (t) is the Lipschitz constant of the function V. Hence,

$$D^+ m(t) \leq \limsup_{h\to 0} \frac{1}{h} L(t+h) M(t+h) \left\| \frac{x(t+h)-x(t)}{h} - f(t,x) \right\|$$

$$+ \limsup_{h\to 0} L(t+h) \left\| \frac{\varphi(t+h,z)-\varphi(t,z)}{h} - \frac{\partial\varphi}{\partial t}(t,z) \right\|$$

$$+ D^+ V(t,\varphi(t,x(t))) \leq 0 + 0 + g(t,m(t))$$

for $t \geq t_0$ and where $M(t)$ is the Lipschitz constant of the function φ and $z = x(t) + hf(t,x)$.
An application of Theorem 1.4.1 of [7] yields,

$$V(t,\varphi(t,x(t,t_0,x_0))) \leq r(t,t_0,u_0) \qquad \text{for} \quad t \geq t_0 .$$

Our result includes the well-known comparison theorem as a special case if we take $\varphi(t,x(t)) \equiv x(t)$. It also includes the *kth* degree comparison theorem of [1] if we take $\varphi(t,x(t)) = h^k(t)\,x(t)$, with $k \geq 0$ and $h \in \mathscr{G}$. In actual fact even though the result of [1] was proved for $k \geq 1$, the same proof carries over for the case $0 \leq k < 1$.

Another useful version of the generalized comparison theorem is:

THEOREM 2.6. *Let all the hypothesis of Theorem* 2.5 *hold except that instead of* (3) *we have*

$$D^+ V(t,\varphi(t,x(t))) + \Phi(\|\varphi(t,x(t))\|) \leq g(t,\varphi(t,x(t)))$$

for $(t,\varphi(t,x(t))) \in R^+ \times R^n$, *where* $\Phi(r) \geq 0$ *is continuous for* $r \geq 0$, $\Phi(0) = 0$ *and* $\Phi(r)$ *is strictly increasing in* r.

Then $V(t_0,\varphi(t_0,x_0) \leq u_0$ implies

$$V(t,\varphi(t,x(t,t_0,x_0))) + \int_{t_0}^{t} \Phi(\|\varphi(s,x(s))\|)\,ds \leq r(t,t_0,u_0), \qquad t \geq t_0 .$$

§ 3. SUFFICIENT CONDITIONS FOR φ-STABILITY

Employing our main result in § 2 we now investigate sufficient conditions for our new concepts of φ-stability of the trivial solution $x = 0$ of the system (1).

THEOREM 3.1. *Assume that there exist functions* $V(t,y)$ *and* $g(t,u)$ *satisfying the following hypothesis*;

(i) $g \in C(R^+ \times R^+, R)$, $g(t,0) = 0$ *and* $g(t,u)$ *is nondecreasing in* u *for each* $t \in R^+$;

(ii) $V \in C(R^+ \times S_\rho, R^+)$, $V(t,0) = 0$, $V(t,y)$ *is locally Lipschitzian in* y *and there exist* $\alpha \in \mathscr{K}$ *and* $\varphi \in \mathscr{D}$ *such that*

$$\alpha(\|\varphi(t,x)\|) \leq V(t,\varphi(t,x)) \qquad \text{for} \quad (t,\varphi(t,x)) \in R^+ \times S_\rho ;$$

(iii) *For* $(t,\varphi(t,x)) \in R^+ \times S_\rho$,

$$D^+ V(t,\varphi(t,x)) \leq g(t,V(t,\varphi(t,x))) .$$

Then (*a*) the equistability of the trivial solution of (2) implies the φ-equistability of the trivial solution of (1). (*b*) the equi-asymptotic stability of the trivial solution of (2) implies the φ-equi-asymptotic stability of the trivial solution of (1).

Remark 2. The definitions of the various stability properties of the system (2) are as defined in [7].

Proof of Theorem 3.1. (*a*) Let $A > 0$, $0 < \varepsilon < \rho$ and $t_0 \in R^+$ be given, then given $\alpha(A\varepsilon) > 0 \, \exists$ a positive function $\delta = \delta(t_0, \varepsilon)$ continuous in t_0 for each ε, such that $u_0 \leq \delta$ implies

$$u(t, t_0, u_0) < \alpha(A\varepsilon) \qquad t \geq t_0$$

where $u(t, t_0, u_0)$ is any solution of (2). For $\varphi \in \mathscr{D}$ let $u_0 = V(t_0, \varphi(t_0, x_0))$, then by (*ii*) $\exists \delta_1 = \delta_1(t_0, \varepsilon) > 0$ continuous in t_0 for each ε such that

$$\| \varphi(t_0, x_0) \| \leq \delta_1 \quad \text{and} \quad V(t_0, \varphi(t_0, x_0)) \leq \delta$$

hold together. We claim that the trivial solution of (1) is φ-equistable. Suppose not, then $\exists$ a solution $x(t) = x(t, t_0, x_0)$ with $\| \varphi(t_0, x_0) \| \leq \delta_1$ and $t_1 > t_0$ such that

$$\| \varphi(t_1, x(t_1)) \| = A\varepsilon \quad , \quad \| \varphi(t, x(t)) \| \leq A\varepsilon \qquad \text{for } t \in [t_0, t_1] .$$

Now if $A > 0$ is chosen such that $0 < A\varepsilon < \rho$, then $\varphi(t, x) \in S_\rho$ for $t \in [t_0, t_1]$ and hence Theorem 2.5 implies.

$$V(t, \varphi(t, x(t)) \leq r(t, t_0, u_0) , \qquad t \in [t_0, t_1]$$

where $r(t, t_0, u_0)$ is the maximal solution of [2] existing for $t \geq t_0$.

$$\therefore \alpha(A\varepsilon) \leq V(t_1, \varphi(t_1, x(t_1))) \leq r(t_1, t_0, u_0) < \alpha(A\varepsilon)$$

which is a contradiction. Hence (*a*) holds.

(*b*) By hypothesis given $\alpha(A\varepsilon) > 0$, $t_0 \in R^+$, $\exists$ positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \varepsilon)$ such that for $t \geq t_0 + T$,

$$u(t, t_0, u_0) < \alpha(A\varepsilon)$$

whenever $u_0 \leq \delta_0$. Choose $u_0 = V(t_0, \varphi(t_0, x_0))$ then $\exists \hat{\delta}_0 = \hat{\delta}_0(t_0) > 0$ such that $\| \varphi(t_0, x_0) \| \hat{\delta}_0$ and $V(t_0, \varphi(t_0, x_0)) \leq \delta_0$ hold together.

Let $\delta_1 = \delta_1(t_0, \rho)$ and define $\delta^* = \min\{\hat{\delta}_0, \delta_1\}$, then $\| \varphi(t_0, x_0) \| \leq \delta^*$ implies $\| \varphi(t, x(t)) \| < A\varepsilon$ for all $t \geq t_0$ by (*a*) hence Theorem 2.5 implies

$$V(t, \varphi(t, x(t)) \leq r(t, t_0, u_0) \qquad \text{for } t \geq t_0 .$$

Let there be a sequence $\{t_k\}$, $t_k \geq t_0 + T$ and $t_k \to \infty$ as $k \to \infty$ such that

$$\| \varphi(t_k, x(t_k, t_0, x_0)) \| \geq A\varepsilon$$

where $x(t, t_0, x_0)$ is a solution of (1) such that $\| \varphi(t_0, x_0) \| \leq \delta^*$. Then

$$\alpha(A\varepsilon) \leq V(t_k, \varphi(t_k, x(t_k)) \leq r(t_k, t_0, u_0) < \alpha(A\varepsilon)$$

which is a contradiction. Hence $\| \varphi(t, x(t, t_0, x_0)) \| < A\varepsilon$ for $t \geq t_0 + T$.

THEOREM 3.2. *Assume that* $\exists V(t, y)$, $g(t, u)$ *satisfying hypothesis* (i) *and* (iii) *of Theorem 3.1. In addition let* $V \in C(R^+ \times S_\rho, R^+)$, $V(t, 0) = 0$,

$V(t, y)$ *is locally Lipschitzian in* y *and* $\exists \alpha \in \mathcal{K} \times \mathcal{K}$ $\varphi \in \mathcal{D}$ *such that for* $(t, \varphi(t, x)) \in R^+ \times S_\rho$,

$$\alpha(t, \|\varphi(t, x(t))\|) \leq V(t, \varphi(t, x)).$$

Then the equistability of the trivial solution of (2) implies the φ-equi-asymptotic stability of the trivial solution of (1).

Proof. Let $\beta(r) = \alpha(0, r)$ then for $(t, \varphi(t, r)) \in R^+ \times S_\rho$,

$$\beta(\|\varphi(t, x)\|) \leq V(t, \varphi(t, x)). \tag{4}$$

Let $0 < \tau < \rho$, $t_0 \in R^+$. By hypothesis given $\beta(A\tau) > 0$, $t_0 \in R^+$ where $A > 0$ such that $0 < A\tau < \rho$, $\exists \delta = \delta(t_0, \tau) > 0$ continuous in t_0 for each τ such that $u_0 \leq \delta$ implies $u(t, t_0, u_0) < \beta(A\tau)$, $t \geq t_0$. Choose $u_0 = V(t_0, \varphi(t_0, x_0))$, then $\exists \delta_1 = \delta_1(t_0, \tau)$ as in Theorem 3.1 such that $\|\varphi(t_0, x_0)\| \leq \delta_1$ and $V(t_0, \varphi(t_0, x_0)) \leq \delta$ hold together.

Using (4) and proceeding with the proof as in Theorem 3.1 (*a*), the solution $x = 0$ is φ-equistable. Let τ be fixed and suppose $\delta_0 = \delta_1(t_0, \tau)$. Choose $0 < \varepsilon < \tau$ and $t_0 \in R^+$ be given.

If $\|\varphi(t_0, x_0)\| \leq \delta_0$, $\alpha \in \mathcal{K} \times \mathcal{K}$ implies $\exists T(t_0, \varepsilon)$ such that

$$\alpha(t, A\varepsilon) > \sup V(t, \varphi(t, x)) \qquad \text{for} \quad t \geq t_0 + T \qquad \|\varphi(t_0, x_0)\| \leq \delta.$$

Let $\{t_k\}$ be a sequence of Theorem 3.1 (*b*), then whenever $\|\varphi(t_0, x_0)\| \leq \delta$,

$$\alpha(t_k, A\varepsilon) \leq V(t_k, \varphi(t_k))) \leq r(t_k, t_0, u_0) < \beta(A\tau)$$

which is a contradiction since $\alpha(t_k, A\varepsilon) \to \infty$ as $t_k \to \infty$.

Hence $x = 0$ is φ-quasi-equi-asymptotically stable. This along with φ-equistability implies $x = 0$ is φ-equi-asymptotically, stable.

We now state a few results which can be proved using standard arguments modified along the lines of our proofs.

THEOREM 3.3. *Let all the hypothesis of Theorem 3.1. hold. In addition assume that* $\exists b \in \mathcal{K}$ *for the same* $\varphi \in \mathcal{D}$ *such that*

$$V(t, \varphi(t, x)) \leq b(\|\varphi(t, x(t))\|)$$

for $(t, \varphi(t, x) \in R^+ \times S_\rho$.

Then the uniform stability of the trivial solution of (2) implies the φ-uniform stability of the trivial solution of (1).

THEOREM 3.4. *Assume that there exists* $V(t, y)$ *satisfying the following*

(i) $V \in C(R^+ \times S_\rho, R^+)$, $V(t, y)$ *is locally Lipschitzian in* y *and* $\exists \alpha, \beta \in \mathcal{K}$, $\varphi \in \mathcal{D}$ *such that for* $(t, \varphi(t . x)) \in R^+ \times S_\rho$,

$$\alpha(\|\varphi(t, x)\|) \leq V(t, \varphi(t, x)) \leq \beta(\|\varphi(t, x)\|),$$

(ii) $D^+ V(t, \varphi(t, x)) \leq 0$ for $(t, \varphi(t, x)) \in R^+ \times S_\rho$.

Then the trivial solution of (1) is φ-uniformly stable.

THEOREM 3.5. *Let all the hypothesis of Theorem 3.3. hold. Then the uniform asymptotic stability of the solution $u = 0$ of (2) implies the φ-uniform asymptotic stability of the trivial solution of (1).*

Remark 3. Our results in this section include sufficient conditions for the Lyapunov stability of the trivial solution $x = 0$. In view of Remark 1 it also include sufficient conditions for stability with respect to h of degree $k \geq 0$, stability of order p and stability with respect to h of degree $k \geq 0$ and order p. In particular they include some results of [3].

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