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On the existence and uniqueness of solutions of certain boundary value problem

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Analisi matematica. — *On the existence and uniqueness of solutions of certain boundary value problem.* Nota di JAROSŁAW MORCHAŁO, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Scopo di questo lavoro è quello di investigare il problema della esistenza e unicità delle soluzioni di un problema di valori al contorno per un'equazione differenziale o integrodifferenziale con argomento deviato.

INTRODUCION

In this article the existence and uniqueness theorems for a boundary value problem of the de La Vallée-Poussin type for differential or integro-differential equations with deviated argument, not solvable with respect to the highest derivative will be given.

Namely, we shall consider a differential equation

$$(1)\quad y^{(n)}(x) = F[x, y(g_0(x)), \cdots, y(g_m(x)), \cdots, y^{(n-1)}(g_0(x)), \cdots \\ \cdots, y^{(n-1)}(g_m(x)), y^{(n)}(g_1(x)), \cdots, y^{(n)}(g_m(x))] \equiv \\ \equiv F(x, [y], [y'], \cdots, [y^{(n)}] \equiv F(x, u_0, \cdots, u_n), \qquad x \in \langle a, b\rangle,$$

or an integrodifferential equation

$$(2)\quad y^{(n)}(x) = F(x, [y], [y'], \cdots, [y^{(n)}]) + \int_a^b K(x, t, y(g_0(t)), \cdots \\ \cdots, y(g_m(t)), \cdots, y^{(n-1)}(g_0(t)), \cdots, y^{(n-1)}(g_m(t)), y^{(n)}(g_1(t)), \cdots, y^{(n)}(g_m(t)))\, dt \\ \equiv F(x, u_0, \cdots, u_n) + \int_a^b K(x, t, u_0, \cdots, u_n)\, dt, \qquad x \in \langle a, b\rangle,$$

with boundary conditions

$$(3)\qquad y^{(r)}(a_k) = 0, (k = 1, 2, \cdots, p, \qquad r = 0, 1, \cdots, r_k - 1),$$

(*) Nella seduta del 12 aprile 1980.

where

$$a \leq a_1 < a_2 < \cdots < a_p \leq b \,, \qquad 1 \leq p \leq n \,, \quad \sum_{k=1}^{p} r_k = n \,,$$

$$y^{(k)}(z) = \begin{cases} \dfrac{d^k y(z)}{dz^k} & \text{if } z \in \langle a , b \rangle \\ 0 & \text{if } z \notin \langle a , b \rangle \end{cases} \qquad k = 0 , 1 , \cdots , n \,.$$

Notations. We shall denote by,

$B_C(\langle a , b \rangle)$ – the Banach space of all continuous functions with their k-th order ($k = 1 , \cdots , n$) derivatives and with the norm

$$(*) \qquad \| y \|_c = \max \left\{ \frac{n!\, n^n}{(n-1)^{n-1} (b-a)^n} \| y \|_1 \,, \quad \frac{(n-1)!\, n}{(b-a)^{n-1}} \| y \|_1 , \cdots \frac{n}{(n-1)(b-a)} \| y^{(n-1)} \|_1 , \| y^{(n)} \|_1 \right\} ,$$

where $\| y \|_1 = \max \{ | y(x) | : x \in \langle a , b \rangle \}$.

$B_A(\langle a , b \rangle)$ – the class of all functions defined on $\langle a , b \rangle$ for which the $(n-1)$-th derivative is absolutely continuous in this interval thus $y^{(n)}(x)$ exists almost everywhere and is summable in $\langle a , b \rangle$:

$B_B(\langle a , b \rangle)$ – the Banach space whose elements are functions $y \in B_A(\langle a , b \rangle)$ such that $\operatorname{vrai} \sup_{x \in \langle a , b \rangle} | y^{(n)}(x) | < \infty$, and the norm is defined by

$$(**) \qquad \| y \|_B = \max \left\{ \frac{n!\, n^n}{(n-1)^{n-1} (b-a)^n} \| y \|_2 \,, \quad \frac{(n-1)!\, n}{(b-a)^{n-1}} \| y \|_2 , \cdots , \frac{(n-k)!\, n}{k (b-a)^{n-k}} \| y^{(k)} \|_2 , \cdots , \| y^{(n)} \|_2 \right\} ,$$

where $\| y \|_2 = \operatorname{vrai} \max_{x \in \langle a , b \rangle} | y(x) |$.

By the solution of the boundary value problems (1), (3) or (2), (3) in B_B [respect. B_C] we mean the function $y \in B_B(\langle a , b \rangle)$ [respect. $y \in B_C(\langle a , b \rangle)$], satisfying almost everywhere in $\langle a , b \rangle$ [respect. everywhere in $\langle a , b \rangle$] the equations (1) or (2) and the boundary conditions (3).

Assumptions. The following assumptions will play an important role in what follows.

Assumption H_1. Suppose that

1) $F \in C(\langle a , b \rangle \times B_C^{n+1} , (-\infty , \infty)) \cap \mathrm{Lip}_{u_i}(p_i , B_C)$, $(i = 0 , 1 , \cdots , n)$

where $p_0 = \sum_{k=0}^{m} p_k^0 , \cdots , p_{n-1} = \sum_{k=0}^{m} p_k^{n-1} , \quad p_n = \sum_{k=1}^{m} p_k^n ,$

2) $g_i \in C(\langle a, b\rangle, \langle a, b\rangle), g_0(x) \equiv x$, $(i = 0, 1, \cdots, m)$,

3) there exists an $r > 0$ such that

$$K_C = \left[\frac{p_0(n-1)^{n-1}(b-a)^n}{n!\, n^n} + \sum_{k=2}^{n} \frac{p_{k-1}(k-1)(b-a)^{n-k+1}}{(n-k+1)!\, n} + p_n\right] < 1$$

for all

$$x \in \langle a, b\rangle \quad , \quad \| y \|_C \leq r ,$$

where

$$M_C \leq r(1 - K_C) \quad , \quad M_C = \| F(x, [0], \cdots, [0]) \|_1 .$$

Assumption H_2. Suppose that

1) the function $F : \langle a, b\rangle \times B_B^{n+1} \to (-\infty, \infty)$ is measurable with respect to the variable $x \in \langle a, b\rangle$ for all $u_i \in B_B$ $(i = 0, \cdots, n)$, continuous with respect to the variable $u_i \in B_B$ almost everywhere in $\langle a, b\rangle$,

2) the function F is essentially bounded with respect to the variable $x \in \langle a, b\rangle$ for all $u_i \in B_B$,

3) the functions g_1 $(i = 1, \cdots, m)$ are measurable in, $\langle a, b\rangle$ and such that the sets $g_i^{-1}(E) = \{x : x \in \langle a, b\rangle \mid g_i(x) \in E\}$ are measurable for arbitrary set $E \in \langle a, b\rangle$ of measure zero.

4) $F \in \mathrm{Lip}_{u_i}(p_i, B_B)$, $\quad (i = 0, \cdots, n)$,

5) there exists an $r > 0$ such that

$$K_B = \left[\frac{p_0(n-1)^{n-1}(b-a)^n}{n!\, n^n} + \sum_{k=2}^{n} \frac{p_{k-1}(k-1)(b-a)^{n-k+1}}{(n-k+1)!\, n} + p_n\right] < 1$$

for all

$$x \in \langle a, b\rangle \quad , \quad \| y \|_B \leq r ,$$

where

$$M_B \leq r(1 - K_B) \quad , \quad M_B = \| F(x, [0], [0], \cdots, [0]) \|_2 .$$

Assumption H_3. Suppose that

1) conditions 1^0 and 2^0 from the assumption H_1 are satisfied,

2) the function

$$K \in C(a \leq t \leq x \leq b \times B_C^{n+1}, \quad (-\infty, \infty)) \cap \mathrm{Lip}_{u_i}(q_i, B_C),$$

where

$$q_0 = \sum_{k=0}^{m} q_k^0, \cdots, q_{n-1} = \sum_{k=0}^{m} q_k^{n-1} \quad , \quad q_n = \sum_{k=1}^{m} q_n^k, \qquad (i = 0, 1, \cdots, n),$$

3) there exists an $r > 0$ such that

$$\overline{K}_C = K_C + (b-a)\left[q_0 \frac{(n-1)^{n-1}(b-a)^n}{n!\, n^n} + \right.$$

$$\left. + \sum_{k=2}^{n} q_{k-1} \frac{(k-1)(b-a)^{n-k+1}}{(n-k+1)!\, n} + q_n\right] < 1$$

for all

$$x \in \langle a, b\rangle \quad , \quad \|y\|_C \leq r ,$$

where

$$\overline{M}_C \leq r(1 - \overline{K}_C) ,$$

$$\| F(x, [0], \cdots, [0]) \|_1 + \left\| \int_a^b K(x, t, [0], \cdots, [0])\, dt \right\|_1 \leq \overline{M}_C .$$

Assumption H_4. Suppose that

1) conditions 1°, 2° from assumption H_2 and 4° from assumption H_3 are satisfied,

2) the function K is measurable with respect to the variable t $(a \leq t \leq x \leq b)$ for every $x \in \langle a, b\rangle$, $u_i \in B_B(\langle a, b\rangle)$, $(i = 0, 1, \cdots, n)$ and continuous with respect to the variables $u_i \in B_B$ for every $x \in \langle a, b\rangle$ and for almost all t,

3) the function K is essentialy bounded with respect to the variable $x \in \langle a, b\rangle$ for all t $(a \leq t \leq x \leq b)$, $u_i \in B_B$,

4) $K \in \mathrm{Lip}_{u_i}(q_i, B_B)$,

5) there exists an $r > 0$ such that $\overline{K}_B = \overline{K}_C$ for all $x \in \langle a, b\rangle$, $\|y\|_B \leq r$, where

$$\overline{M}_B \leq r(1 - \overline{K}_B), \|F(x, [0], \cdots, [0])\|_2 + \left\| \int_a^b K(x, t, [0], \cdots, [0])\, dt \right\|_2 \leq \overline{M}_B .$$

Main results. Let us write the problem (1), (3) in the equivalent form of an integral equation

$$y(x) = \int_a^b G(x, t)\, F(t, [y], \cdots, [y^{(n)}])\, dt \tag{4}$$

where $G(x, t)$ is the Green function of the differential equation $y^{(n)}(x) = 0$ with boundary conditions (3).

Let A denote the operator defined by the right side of (4) i.e.

$$(5)\qquad (Ay)(x) = \int_a^b G(x, t)\, F(t, [y], \cdots, [y^{(n)}])\, dt .$$

THEOREM 1. *Suppose that assumption* H_1 *hold. Then the problem* (1), (3): *has a unique solution on the set* $K_0 = \{y(x) : y(x) \in B_C(\langle a, b\rangle), \|y\|_C \le r\}$. *This solution is the limit of the sequence of Picard's successive approximations.*

$$y_0(x) = \int_a^b G(x, t)\, F(t, [0], \cdots, [0])\, dt ,$$

$$y_k = Ay_{k-1}, \qquad (k = 1, 2, \cdots).$$

Proof. We have to prove that operator A has the contraction property. By (5), assumption H_1 and Lemma from [2] we have for $y, z \in K_0$:

$$\|(Ay)(x) - (Az)(x)\|_1 \le \frac{(n-1)^{n-1}(b-a)^n}{n!\, n^n} K_C \|y - z\|_C ,$$

$$\|(Ay)'(x) - (Az)(x)\|_1 \le \frac{(b-a)^{n-1}}{(n-1)!\, n} K_C \|y - z\|_C ,$$

. . . .

$$\|(Ay)^{(n-1)}(x) - (Az)^{(n-1)}(x)\|_1 \le \frac{(n-1)(b-a)}{n} K_C \|y - z\|_C ,$$

$$\|(Ay)^{(n)}(x) - (Az)^{(n)}(x)\|_1 \le K_C \|y - z\|_C .$$

Hence

$$(6)\qquad \|(Ay)(x) - (Az)(x)\|_C \le K_C \|y - z\|_C .$$

Moreover

$$\|(A0)(x)\|_1 \le M_C \frac{(n-1)^{n-1}(b-a)^n}{n!\, n^n},$$

$$\|(A0)'(x)\|_1 \le M_C \frac{(b-a)^{n-1}}{(n-1)!\, n},$$

. . . .

$$\|(A0)^{(n-1)}(x)\|_1 \le M_C \frac{(n-1)(b-a)}{n},$$

$$\|(A0)^{(n)}(x)\|_1 \le M_C .$$

Hence

$$\| (A0)(x) \|_C \leq r(1 - K_C) . \tag{7}$$

New theorem 1 is implied by the fixed-point theorem given in [3].

In the same way we prove the following theorems.

THEOREM 2. *Let assumption* H_2 *hold. Then the problem* (1), (3) *has a unique solution in* $K_0 = \{y(x) : y(x) \in B_B(\langle a, b \rangle), \| y \|_B \leq r\}$.

THEOREM 3. *Assume that* H_3 *hold. Then the problem* (2), (3) *has a unique solution in* K_0.

THEOREM 4. *If assumption* H_4 *is satisfied, then the problem* (2), (3) *has a unique solution in* K_0.

Remark. From Theorems 1 and 2 (in the case $n = 2$) we get Theorems 1 and 3 given in the paper by G. A. Kamienski and A. D. Myszkis [1].

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