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On the existence of torsion-free modules with a prescribed type-set

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Algebra. — *On the existence of torsion-free modules with a prescribed type-set.* Nota di LUCIE DE MUNTER-KUYL, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si costruisce un A-modulo (A dominio principale arbitrario) con prescritto insieme di tipi, e si mostra come si possono ottenere moduli completamente anisotropi.

Let A be a principal domain and let K be its field of fractions. Let P be a complete set of pairwise non-associated irreducible elements of A (i.e. P contains one and only one generator of each non-zero prime ideal of A), let v_p be the valuation of K associated with $p \in P$, and let K_p be the completion of K with respect to v_p.

We define a *superdivisor of* A (i.e. a "characteristic", in abelian group theory) to be any mapping $\mu : P \to \mathbf{N} \cup \{\infty\}$. The *type* $\tau(\mu)$ of a superdivisor μ is defined as usual, and we refer the reader to [1] for the properties of superdivisors, types, and related concepts. If μ and μ' are two superdivisors, we denote their GCD (or meet) by $[\mu, \mu']$. Let $\mathcal{S}$ be a set of superdivisors. As in [2], we shall say that $\mathcal{S}$ is *relatively disjoint* if there exists a superdivisor $\mu_0 \in \mathcal{S}$ such that $[\mu, \mu'] = \mu_0$, for every $\mu, \mu' \in \mathcal{S}$, $\mu \neq \mu'$. We say that a set of types is relatively disjoint if it can be represented by a relatively disjoint set of superdivisors. If M is a rank two torsion-free A-module, we denote its type-set by T(M). Thus, $T(M) = \{\tau(N)\}$, where N runs through all rank one pure submodules of M and where $\tau(N)$ denotes the type of N.

THEOREM 1. *Let* T *be a relatively disjoint set of types. There exists a rank two torsion-free* A-*module* M *such than* T(M) = T.

Proof. Let $\mathcal{S}$ be a relatively disjoint set of superdivisors representing T, let μ_0 denote the GCD of any two distinct elements of $\mathcal{S}$, and let $\tau_0 = \tau(\mu_0)$.

If card $\mathcal{S} \leq 3$, the desired module clearly exists; it can even be chosen so as to be decomposable.

Now, let card $\mathcal{S} > 3$. Choose $\mu_1, \mu_2 \in \mathcal{S}$, with μ_0, μ_1, μ_2 all distinct, and set $\mathcal{S}^* = \mathcal{S} - \{\mu_0, \mu_1, \mu_2\}$. For every $\mu \in \mathcal{S}^*$, let $P_\mu = \{p \in P ; \mu(p) > \mu_0(p)\}$, let $P^* = \bigcup_{\mu \in \mathcal{S}^*} P_\mu$, and let $P_\mu^\infty = \{p \in P_\mu ; \mu(p) = \infty\}$. Because $\mathcal{S}$ is relatively disjoint, we have $P_\mu \neq \varnothing$ for every $\mu \in \mathcal{S}^*$, whereas $P_\mu \cap P_{\mu'} = \varnothing$ if $\mu \neq \mu'$, and we have $\mu_0(p) = \mu_1(p) = \mu_2(p) < \infty$ for all $p \in P^*$,

We note that the preceding relations guarantee that card $\mathcal{S} \leq$ card $P + 1$.

(*) Nella seduta del 12 aprile 1980.

This, in turn, ensures that we can, and do, choose an injective mapping $\Psi : \mu \mapsto k_\mu$ from $\mathcal{S}^* \cup \{\mu_0\}$ to $K^* = K - \{0\}$ satisfying

$$(1) \qquad v_p(k_\mu) = 0 \qquad \text{for all} \quad p \in P_\mu .$$

Define the superdivisor $\tilde{\mu}$ by $\tilde{\mu}(p) = \infty$ if $p \in P^*$, and $\tilde{\mu}(p) = 0$ otherwise.

For each $p \in P^*$, choose $\rho(p) \in K_p$ satisfying

(*a*) $\rho(p) = k_\mu$, if $p \in P_\mu^\infty$;

(*b*) $\rho(p) \in K_p - K$, $v_p(\rho(p)) = 0$ and $v_p(\rho(p) - k_\mu) = \mu(p) - \mu_0(p)$, if $p \in P_\mu - P_\mu^\infty$ (Observe that for any μ in $\mathcal{S}^*$ and k in K^*, with $k \neq k_\mu$, we thus have $v_p(\rho(p) - k) = 0$ for almost all $p \in P_\mu$). Then, for each $\rho(p)$, select a sequence $(a_n(p))$ of elements of $A - (p)$, converging to $\rho(p)$, and such that $v_p(\rho(p) - a_n(p)) = n + \mu_0(p)$.

For $p \notin P^*$, choose

(*c*) $\rho(p) = 0$, if $\mu_1(p) + \mu_2(p) = \infty$ and

(*d*) $\rho(p) = p^{\mu_1(p) - \mu_2(p)}$, if $\mu_1(p) + \mu_2(p) < \infty$.

Finally, set $\rho = (\rho(p)) \in \prod_{p \in P} K_p$.

Now, let x_1, x_2 be two independent elements of the A-module K^2 and let M be the A-submodule generated by the set $\{p^{-r} x_1, p^{-s} x_2$, for all $p \in P$ and $r, s \in \mathbf{N}, r \leq \mu_1(p), s \leq \mu_2(p)\} \cup \{p^{-(n+\mu_0(p))}(x_1 + a_n(p) x_2)$, for all $p \in P^*$ and $n \in \mathbf{N}\}$. Then, using the same notations as in [1], we have $\operatorname{inv}(M, x_1, x_2) = (\mu_1, \mu_2, \tilde{\mu}, \bar{\rho})$ and it was proved in [1] (Corollary 3.3) that the module M has T for type-set. More precisely, denoting by $N(k)$, $k \in K^*$, the rank one pure submodule $K(x_1 + kx_2) \cap M$, we showed that $\tau(N(k_\mu)) = \tau(\mu)$ for all $\mu \in \mathcal{S}^*$ and $\tau(N(k)) = \tau_0$ for all $k \in K^* - \Psi(\mathcal{S}^*)$. In addition, the pure submodules generated by x_1 and x_2 have types $\tau(\mu_1)$ and $\tau(\mu_2)$ respectively.

Remark. When choosing the domain of Ψ, we added μ_0 to $\mathcal{S}^*$ to make sure that the type τ_0 be actually represented by some rank one pure submodule of M. In case Ψ is surjective, $N(k_{\mu_0})$ is the only rank one pure submodule of type τ_0 and M is then completely anisotropic.

The requirement that μ_0 be in $\mathcal{S}$ can be dropped from the definition of a relatively disjoint set of superdivisors. In this case, when Ψ is bijective it gives rise to a completely anisotropic module in which the type τ_0 is no more represented. The problem of proving the existence of a completely anisotropic module having T for type-set is thus reduced to that of building a bijection Ψ from $\mathcal{S} - \{\mu_1, \mu_2\} = \mathcal{S}^* \cup \{\mu_0\}$ (or $\mathcal{S}^*$, depending on whether μ_0 belongs or does not belong to $\mathcal{S}$) onto K^*, and which would satisfy (1). This, of course, will not be possible unless card $\mathcal{S}$ = card P = card K = λ, where λ has to be an infinite cardinal.

We want to mention here that Ito's construction is also based on such a bijection (see [2], (**)) and we shall see that it does not depend as much as it seems to on the particular relationship between divisibility by prime numbers and the natural order on $\mathbf{N}$, i.e. on the "distribution of prime numbers". Except for a slight requirement, any well-ordering of P will do.

THEOREM 2. *Let* T *be a relatively disjoint set of types and suppose card* T = card P = card K. *Then, there exists a completely anisotropic rank two torsion-free* A-*module* M *such that* T (M) = T.

Proof. Keeping in mind the proof of Theorem 1, and assuming for simplicity that $\mu_0 \notin \mathcal{S}$, we need only build a bijection $\Psi : \mu \mapsto k_\mu$ from $\mathcal{S}^*$ onto K^*, such that $v_p(k_\mu) = 0$ for all $p \in P_\mu$.

The hypothesis relative to the cardinalities can be restated

$$\text{card } \mathcal{S}^* = \text{card P} = \text{card } K^* = \lambda \,, \tag{2}$$

where λ is infinite and a limit ordinal.

On P, consider any well-ordering allowing to write $P = \{p(\alpha)\}_{\alpha<\lambda}$. Set $p_\mu = \inf P_\mu$.

The well-ordering of P induces one on its subset $\{p_\mu\}_{\mu \in \mathcal{S}^*}$, and therefore on $\mathcal{S}^*$. As λ is the smallest ordinal in bijective correspondence with P, by (2) we must have $\mathcal{S}^* = \{\mu(\alpha)\}_{\alpha<\lambda}$ and $\{p_\mu\}_{\mu\in\mathcal{S}^*} = \{p_{\mu(\alpha)}\}_{\alpha<\lambda}$. Note that $p(\alpha) \preceq p_{\mu(\alpha)}$ for any ordinal $\alpha < \lambda$.

Denote the group of units of A by U and let $\xi = \text{card U}$. If ξ is infinite, then it is a limit ordinal and, anyway, we have $\xi \leq \lambda$. Well-order U so as to have $U = \{u(\beta)\}_{\beta<\nu}$, where $\nu = \xi + 1$, if ξ is finite, and $\nu = \xi$, if ξ is a limit ordinal.

It remains to well-order K^*, which we do in the following manner:

(i) $K^* = \{k(\alpha)\}_{\alpha<\lambda}$;

(ii) for any pair of ordinals α and γ such that $\gamma \leq \alpha < \lambda$, $p(\alpha)$ does not divide $k(\gamma)$;

(iii) for any limit ordinal $\alpha \leq \lambda$, and any choice of $\alpha_0 < \inf(\alpha, \nu)$, $n \in \mathbf{N}^*$, $r_1, \cdots, r_n \in \mathbf{Z}$ and $\alpha_1, \cdots, \alpha_n < \alpha$, there exists an ordinal $\beta < \alpha$ such that $k(\beta) = u(\alpha_0)\, p(\alpha_1)^{r_1} \cdots p(\alpha_n)^{r_n}$ (this is clearly possible since the cardinality of the set of all such $k(\beta)$ is equal to card α).

The ensuing bijection $\Psi : \mu(\alpha) \mapsto k(\alpha)$ has the desired property. This completes the proof.

Now suppose A is a Dedekind domain, not necessarily principal. Then P has to be replaced by the set $\mathcal{P}$ of all non-zero prime ideals $\mathfrak{p}$ of A. If T is a relatively disjoint set of types and $\mathcal{S}$ is any relatively disjoint set of superdivisors representing T, one cannot guarantee anymore the existence of k_μ in K^*, with the preassigned value zero for $v_\mathfrak{p}(k_\mu)$ whenever $\mu(\mathfrak{p}) > \mu_0(\mathfrak{p})$. However, if $\mathcal{P}_\mu^\infty = \varnothing$ for all $\mu \in \mathcal{S}^*$, it will be possible to select a suitable set $\mathcal{S}$. Indeed, let $\Psi : \mu \mapsto k_\mu$ be any injective mapping from $\mathcal{S}$ to K^*. Since $v_\mathfrak{p}(k_\mu) = 0$ for almost all $\mathfrak{p} \in \mathcal{P}$, we can replace μ (after possibly changing the value of $\mu(\mathfrak{p})$ at a finite number of places where $\mu(\mathfrak{p}) < \infty$) by a superdivisor $\overline{\mu}$, of the same type, and such that $\overline{\mu}(\mathfrak{p}) = \mu_0(\mathfrak{p})$ when $v_{\mathfrak{p}}(k_\mu) \neq 0$. This gives rise to a relatively disjoint set $\overline{\mathscr{S}}$, which satisfies (1) and is also representing T. Then, with only minor modifications to the proof of Theorem 1 (e.g. in the choice of ρ), we obtain the following.

THEOREM 3. *Let* A *be a Dedekind domain and let* $\mathscr{P}$ *be the set of all non-zero prime ideals of* A. *Let* $\mathscr{S}$ *be a relatively disjoint set of superdivisors and let* μ_0 *be the* GCD *of any two distinct elements of* $\mathscr{S}$. *Suppose* $\mathscr{S}$ *contains at most two superdivisors* μ *such that* $\mu_0(\mathfrak{p}) < \mu(\mathfrak{p}) = \infty$ *for some* $\mathfrak{p} \in \mathscr{P}$. *Then, there exists a rank two torsion-free* A*-module* M *such that* $T(M) = \{\tau(\mu)\,;\, \mu \in \mathscr{S}\}$.

REFERENCES

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[2] RYUICHI ITO (1975) – *On type-sets of torsion-free abelian groups of rank 2*, « Proc. Amer. Math. Soc. », *48*, 39–42.