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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On first Čech groups* H^0, H^1 *of maximal ideal spaces.* Nota di EDOARDO BALLICO, ARTURO V. FERREIRA e PIER DANIELE NAPOLITANI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si stabilisce un rapporto fra i primi gruppi di coomologia dello spazio strutturale e la struttura algebrica di un'algebra topologica commutativa.

1. This note and [1] are the first of a series on the use of algebraic topology in the study of general topological algebras with various applications to complex analysis; next we will treat subjects involving the Chern character and Picard groups.

A will denote a complex unitary complete topological algebra whose topology can be defined by a system of algebra seminorms. Suppose $\mathcal{N}$ is a filtering system of algebra seminorms which defines the topology of A. For each $p \in \mathcal{N}$ we denote by A_p the completion of the normed algebra (A/ker$p$, p/ker$p$), $\hat{p}$ the norm on A_p, and by π_p the algebra morphism $A \to A_p$ that is obtained by composing the canonical epimorphism $A \to A/\ker p$ and the natural injection $A/\ker p \to A_p$. We have $\ker \pi_p = \ker p$ and, if $q \in \mathcal{N}$ is finer than p, the natural mapping $A/\ker q \to A/\ker p$ extends as a (continuous) algebra morphism $\pi_{pq} : A_q \to A_p$. As we will have $\pi_p = \pi_{pq} \circ \pi_q$, the system (A_p, π_{pq}) gives rise to a projective limit with which A can be identified because A is complete.

(*) Nella seduta del 12 aprile 1980.

Let A^* denote the group of invertible elements in A. When A^* is known to be open (which is the case if A is Banach), if $p \in \mathcal{N}$ is a seminorm for which some p-ball centered at 1 is contained in A^*, then for each $a \in A$, $sp_A(a)$ is a compact subset of $\mathbf{C}$ equal to $sp_{A_p}(a)$ and therefore $a \in A^*$ whenever the spectral radius of $a - 1$ in A_p is < 1; in particular, the open p-ball centered at 1 of radius 1 is contained in A^*.

In the sequel of the present note A will be supposed moreover commutative. Each closed maximal ideal is the kernel of a continuous complex character on A by the Gel'fand–Mazur theorem, so that the weak*—dual A' induces on the set $\Sigma(A)$ of closed maximal ideals a topology γ (the Gel'fand topology) which is the coarser one that renders continuous Gel' fand transforms $\hat{a}$ of elements a in A. On the other hand, the transposed mappings ${}^t\pi_p : A'_p \to A'$ are injective and therefore, if we identify each $\Sigma(A_p)$, which is compact for its Gel'fand topology, with its image in $\Sigma(A)$ by ${}^t\pi_p$, we will obtain $\Sigma(A) = \bigcup_{p \in \mathcal{N}} \Sigma(A_p)$. This enables us to consider on $\Sigma(A)$ another useful topology λ finer, and in general different from Gel'fand topology—the inductive limit topology of the compact spaces $\Sigma(A_p)$. When A^* is open in A (which is always the case if A is barrelled and $\Sigma(A)_\gamma$ is compact), there is p in $\mathcal{N}$ for which we have $\Sigma(A)_\gamma = \Sigma(A)_\lambda = \Sigma(A_p)_\gamma$.

The Gel' fand transform ^ is a (unitary) algebra morphism $A \to \mathscr{C}(\Sigma(A))$ when $\Sigma(A)$ is given the γ or λ topology. $\mathscr{C}(\Sigma(A)_\lambda)$ is a complete algebra when endowed with the topology of uniform convergence on the compact sets $\Sigma(A_p)$, $p \in \mathcal{N}$, and ^ is then continuous. On the algebra $\mathscr{C}(\Sigma(A)_\gamma)$ will be not considered any topology; however it contains the pointwise limit e^f of the exponential series for any $f \in \mathscr{C}(\Sigma(A)_\gamma)$.

Now, we turn to describe $H^0(\Sigma(A))$. This group is intimately connected to the Boolean structure of open-closed subsets of $\Sigma(A)_\gamma$, $\Sigma(A)_\lambda$, and so must be closely related to the system I of idempotents of A. What we are doing is just to illuminate this point..

Consider the group homomorphism $\exp : A \to A^*$ which sends a into $e^{2\pi i a}$, its kernel E and its image U. We have clearly $U \subset A^1$, the connected component of 1 in A^* which is thus a closed subgroup of A^*. If A reduces to the complex number field we have $U = A^1 = A^*$; when A^* is open in A, A^* is locally connected, and by using the logaritmic series we see at once that U is open in A^* which implies $U = A^1$. In general however, A^1 is not open in A^*, U is not open in A^1 and we may have $U \neq A^1$. For example, in the product algebra $A = \mathscr{C}(\mathbf{T})^{\mathbf{N}}$, $\mathbf{T}$ the unidimensional torus, we have A^* not open in A, $U = A^1$ not open in A^*. In the algebra $A = \mathscr{C}(\mathbf{T})$ endowed with the topology of uniform convergence on convergent sequences, $U \neq A^1$ and is dense in $A^1 = A^*$.

The discussion of the groups U, A^1 will be continued in paragraph 2; for the time being we are mainly interested in the group E.

First we observe that every element in E has an integer-valued Gel'fand transform and the intersection of E with the radical R(A) (which is equal to the kernel of the Gel'fand transform ^) reduces to 0. This implies that E is a discrete subset of A. In particular, every convergent sequence in E must be constant from a certain point on.

It is clear that $E \supset I$ and because E can be considered as a **Z**-module, every finite linear combination with integer coefficients of elements of I belongs to E. Let us establish the converse.

Consider an element e in E; its Gel'fand transform $\hat{e}$ is integer-valued so that, if we denote for each $n \in \mathbf{Z}$ by γ_n the circle centered at n with radius $1/4$ in the complex plane oriented as usual, the integral $(1/2\pi i)\int_{\gamma_n}(\lambda - e)^{-1}\,d\lambda$ exists in A and we shall represent its values by $j_n(e)$. Fix n_1, n_2 in **Z**; by a straightforward use of the analytic calculus we recognize that for each $p \in \mathcal{N}$,

$$\pi_p(j_{n_1}(e)\,j_{n_2}(e)) = \left[(1/2\pi i)\int_{\gamma_{n_1}}(\lambda - \pi_p(e))^{-1}\,d\lambda\right]\left[(1/2\pi i)\int_{\gamma_{n_2}}(\lambda - \pi_p(e))^{-1}\,d\lambda\right]$$

is 0 when $n_1 \neq n_2$ and equals $\pi_p(j_{n_1}(e))$ if $n_1 = n_2$. Hence $j_{n_1}(e)$ is an idempotent in A which is orthogonal to every $j_{n_2}(e)$ with $n_1 \neq n_2$. Moreover, since given $p \in \mathcal{N}$, $\int_{\gamma_n}(\lambda - \pi_p(e))^{-1}\,d\lambda$ is zero whenever $n \notin sp_{A_p}(\pi_p(e))$, we can conclude that $p(j_n(e)) = 0$ for $n \notin sp_{A_p}(\pi_p(e))$ which means just that for every $p \in \mathcal{N}$, $p(j_n(e)) = 0$ except for finitely-many n in **Z**! There follows that the series $\sum_{n\in\mathbf{Z}} n\,j_n(e)$, $\sum_{n\in\mathbf{Z}} j_n(e)$ are absolutely convergent in A. We have clearly $\sum_{n\in\mathbf{Z}} n\,j_n(e)$, $\sum_{n\in\mathbf{Z}} j_n(e) \in E$ and $\left(\sum_{n\in\mathbf{Z}} n\,j_n(e)\right)^{\wedge} = \hat{e}$, $\left(\sum_{n\in\mathbf{Z}} j_n(e)\right)^{\wedge} = \hat{1}$ which means actually that $\sum_{n\in\mathbf{Z}} n\,j_n(e) = e$ and $\sum_{n\in\mathbf{Z}} j_n(e) = 1$; such a decomposition of e is obviously unique. We have thus proved the following generalisation of a known result of Banach algebra theory:

LEMMA 1. E *contains the sum of every absolutely convergent series of integral multiples of idempotents. Each element e of* E *is the sum of a uniquely determined series* $\sum_{n\in\mathbf{Z}} n\,j_n(e)$ *of integer multiples of pairwise orthogonal idempotents with sum* 1. *If on* A *exists a continuous norm, then only finitely-many* $j_n(e)$ *are* $\neq 0$.

We are now in a position to prove

THEOREM 1. *We have* $H^0(\Sigma(A)_\gamma, \mathbf{Z}) = H^0(\Sigma(A)_\lambda, \mathbf{Z})$ *and* E *is naturally isomorphic to these groups.*

Proof. The Gel'fand transform of an $e \in E$ is a convergent series with integer coefficients of characteristic functions of closed—open subsets of $\Sigma(A)$ with disjoint supports and so we have a natural homomorphism $E \to H^0(\Sigma(A)_\gamma, \mathbf{Z})$; we shall denote by ι its composition with the inclusion $H^0(\Sigma(A)_\gamma, \mathbf{Z}) \to H^0(\Sigma(A)_\lambda, \mathbf{Z})$. Let us verify that ι is onto.

Take an element of $H^0(\Sigma(A)_\lambda, \mathbf{Z})$, we can associate with it in a stardard way an integer-valued continuous function θ; we claim just that $\theta = \hat{e}$ for some e in E. Fix n in $\mathbf{Z}$ and for each $p \in \mathcal{N}$ denote by j_n^p the unique idempotent in A_p for which we have according to Šilov idempotent's theorem, that $(j_n^p)^\wedge$ is the characteristic function of $\theta^{-1}(n) \cap \Sigma(A_p)$. We also must have $\pi_{pq}(j_n^q) = j_n^p$ for $\pi_{pq}(j_n^q)^\wedge = (j_n^p)^\wedge$, whenever q is finer than p. It follows that there is j_n in I for which $\pi_p(j_n) = j_n^p$, $p \in \mathcal{N}$. Now, the series $\sum_{n \in \mathbf{Z}} n j_n$ converges in A because for fixed $p \in \mathcal{N}$ only finitely-many $p(j_n)$ are $\neq 0$ and so its sum is the claimed $e \in E$.

2. This paragraph concerns the group $H^1(\Sigma(A), \mathbf{Z})$. First we prove

PROPOSITION 1. $\hat{U} = \exp(\mathscr{C}(\Sigma(A)_\lambda)) \cap \hat{A}$; *moreover, if* $f \in \mathscr{C}(\Sigma(A)_\lambda)$ *satisfies the equation* $\hat{a} = \exp(f)$ *for some* $a \in A$, *there exists a unique* $b \in A$ *such that* $a = \exp(b)$ *and* $\hat{b} = f$. *Therefore the Gel'fand transform establishes an isomorphism into*

$$A^*/U \to \mathscr{C}(\Sigma(A)_\lambda)^*/\exp(\mathscr{C}(\Sigma(A)_\lambda)) .$$

PROPOSITION 1'. U *is dense in* A^1 *and* $A^1 = \varprojlim(\exp(A_p), \pi_{pq})$. *Also* $(A^1)^\wedge = \mathscr{C}(\Sigma(A)_\lambda)^1 \cap \hat{A}$. *Gel'fand transform induces an isomorphism into*

$$A^*/A^1 \to \mathscr{C}(\Sigma(A)_\lambda)^*/\mathscr{C}(\Sigma(A)_\lambda)^1 .$$

Proof of proposition 1. The uniqueness part is clear from the fact zero is the unique solution of $\exp(x) = 1$ in R(A). To prove the existence it suffices for each $p \in \mathcal{N}$ to solve the equation $\pi_p(a) = \exp(b_p)$, $\hat{b}_p = f_{|\Sigma(A_p)}$ in A_p, which is possible by using analytic calculus, and observe that $(b_p)_{p \in \mathcal{N}}$ belongs to $\varprojlim A_p$.

Proof of proposition 1'. We first show that U is dense in A^1. Let $a \in A^1$, $p \in \mathcal{N}$ and δ be a real number > 0. $\pi_p(A^1) \subset A_p^1 = \exp(A_p)$ because $\pi_p(A^1)$ must be connected and contains 1. Hence $\pi_p(a) = \exp(b_p)$ with $b_p \in A_p$, and by choosing some b in A whith $\dot{p}(\exp(\pi_p(b)) - \exp(b_p)) < \delta$, which is possible because $\pi_p(A)$ is dense in A_p, we have in A, $p(\exp(b) - a) < \delta$. The argument also establishes the first equality relation. $(A^1)^\wedge \subset \mathscr{C}(\Sigma(A)_\lambda)^1$ is clear; the converse inclusion is a consequence of the following approximation lemma:

LEMMA 2. *Let* $a \in A^*$ *be such that* $\hat{a}$ *is in the closure* $\mathscr{C}(\Sigma(A)_\lambda)^1$ *of* $\exp(\mathscr{C}(\Sigma(A)_\lambda))$; *then* a *is in the closure* A^1 *of* U.

Proof. It is enough to observe that for each $p \in \mathcal{N}$, $\hat{a}_{|\Sigma(A_p)} = \pi_p(a)^\wedge \in \mathscr{C}(\Sigma(A_p))^1 = \exp(\mathscr{C}(\Sigma(A_p)))$ and then apply an argument similar to the first part of the proof of proposition 1'.

Let τ be one of the topologies γ or λ, denote by $\mathscr{C}_\tau$ the sheaf of germs of continuous functions and by $\mathscr{C}_\tau^*$ the sheaf of germs of continuous invertible functions on $\Sigma(A)_\tau$.

The commutative diagram of exact sequences of sheaves

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z} & \to & \mathscr{C}_\gamma & \xrightarrow{\exp} & \mathscr{C}^*_\gamma & \to & 0 \\ & & \| & & \cap\downarrow & & \cap\downarrow & & \\ 0 & \to & \mathbf{Z} & \to & \mathscr{C}_\lambda & \xrightarrow{\exp} & \mathscr{C}^*_\lambda & \to & 0 \end{array}$$

implies a commutative diagram of cohomology sequences

$$\begin{array}{ccccccccc} 0 \to H^0(\Sigma(A)_\gamma, \mathbf{Z}) & \to & \mathscr{C}(\Sigma(A)_\gamma) & \xrightarrow{\exp} & \mathscr{C}(\Sigma(A)_\gamma)^* & \to & H^1(\Sigma(A)_\gamma, \mathbf{Z}) & \xrightarrow{\nu_\gamma} & H^1(\Sigma(A)_\gamma, \mathscr{C}_\gamma) \\ \wr\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 \to H^0(\Sigma(A)_\lambda, \mathbf{Z}) & \to & \mathscr{C}(\Sigma(A)_\lambda) & \xrightarrow{\exp} & \mathscr{C}(\Sigma(A)_\lambda)^* & \to & H^1(\Sigma(A)_\lambda, \mathbf{Z}) & \xrightarrow{\nu_\lambda} & H^1(\Sigma(A)_\lambda, \mathscr{C}_\lambda) \end{array}$$

which tells us that $H^0(\Sigma(A)_\tau, \mathbf{Z})$ is just the kernel of the considered exponential function whereas $\ker \nu_\tau$ is its cokernel. In particular, if $H^1(\Sigma(A)_\tau, \mathscr{C}_\tau)$ vanishes, we have $\mathscr{C}^*(\Sigma(A)_\tau)/\exp(\mathscr{C}(\Sigma(A)_\tau)) = H^1(\Sigma(A)_\tau, \mathbf{Z})$; this is mainly the case whenever $\Sigma(A)_\tau$ is paracompact because $\Sigma(A)_\tau$ being also completely regular, the sheaf $\mathscr{C}_\tau$ is soft.

By applying the classical H^1-theorem of Arens and Royden for Banach algebras it is now easy to draw a lot of consequences from the information which is contained in the above diagrams. Here we will only explicitate what can be said in general without further hypothesis on A or $\Sigma(A)$. In another paper Fréchet and Schwartz algebras will be considered.

THEOREM 2. *We have a commutative diagram of injective homomorphisms*

$$\begin{array}{ccc} A^*/A^1 & \to & \varprojlim A^*_p/\exp(A_p) \\ \hat{}\downarrow & & \hat{}\downarrow\wr \\ \mathscr{C}(\Sigma(A)_\lambda)^*/\mathscr{C}(\Sigma(A)_\lambda)^1 & \to & \varprojlim \mathscr{C}(\Sigma(A_p))^*/\exp(\mathscr{C}(\Sigma(A_p))) \end{array}$$

and $\varprojlim A^*_p/\exp(A_p) \xrightarrow{\simeq} \varprojlim H^1(\Sigma(A_p), \mathbf{Z})$.

THEOREM 2'. *Let* $\Sigma(A)^\tau$ *be the maximal ideal space of the algebra of continuous bounded functions on* $\Sigma(A)_\tau$ *with the uniform norm. Then the group* $\mathscr{C}(\Sigma(A)_\tau)^*/\exp(\mathscr{C}(\Sigma(A)_\tau))$ *is isomorphic to the image of the natural homorphism* $H^1(\Sigma(A)^\tau, \mathbf{Z}) \to H^1(\Sigma(A)_\tau, \mathbf{Z})$.

REFERENCES

[1] A. V. FERREIRA–P. D. NAPOLITANI (1980) – *On invertible holomorphic functions with values in a topological algebra*, «Rend. Acc. Naz. Lincei», *68*.