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Theorems of the Cartan-Thullen Type and θ-envelope of Holomorphy for Every Holomorphy Type θ

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Theorems of the Cartan-Thullen Type and* θ*-envelope of Holomorphy for Every Holomorphy Type* θ. Nota di LUIZA A. MORAES, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Dopo aver enunciato alcuni teoremi di tipo Cartan-Thullen sugli aperti c–olomorficamente convessi e cb–olomorficamente convessi, si costruiscono i θ–inviluppi di olomorfia per ogni tipo di olomorfia θ.

1. THE CARTAN-THULLEN THEOREM

DEFINITION 1.1. An open set $U \subset E$ is *c-holomorphically convex*

$$\text{if} \quad \hat{K}_c = \{t \in U \,;\, |f(t)| \leq \|f\|_k \quad \forall f \in \mathcal{H}_c(U)\}$$

is U-bounded for all compact $K \subset U$. An open set $U \subset E$ is *cb-holomorphically convex* if $\hat{B}_{cb} = \{t \in U \,;\, |f(t)| \leq \|f\|_B \ \forall f \in \mathcal{H}_{cb}(U)\}$ is U-bounded for all U-bounded sets $B \subset U$.

DEFINITION 1.2. We say that $f \in \mathcal{H}_c(U)$ cannot be extended to a compactly holomorphic function in a neighbourhood of $\xi \in \bar{U}$ if it is impossible to find two connected open sets U_0, V such that: (*i*) $U \cap V \supset U_0 \neq \varnothing$, $\xi \in V \not\subset U$, (*ii*) $\exists f_1 \in \mathcal{H}_c(V)$ such that $f_1|_{U_0} = f|_{U_0}$.

(*) Nella seduta dell'8 marzo 1980.

DEFINITION 1.3. We say that $f \in \mathscr{H}_{cb}(U)$ cannot be extended to a compactly holomorphic function of U-bounded type in a neighbourhood of $\xi \in \bar{U}$ if it is impossible to find two connected open sets U_0, V such that: (*i*) $U \cap V \supset U_0$ $U_0 \neq \varnothing$, $\xi \in V \not\subset U$; (*ii*) $\exists f_1 \in \mathscr{H}_{cb}(V)$ such that $f_1/U_0 = f/U_0$.

THEOREM 1.1. *Let* U *be a connected open subset of the Banach space* E *such that each* $\xi \in \partial U$ *has a fundamental system of neighbourhoods* $N_n(\xi)$ *such that* $U \cap N_n(\xi)$ *is connected for each n. Then the following properties are equivalent*:

a) *For each* $\xi \in \partial U$ *there exists* $f \in \mathscr{H}_c(U)$ *which cannot be extended to a compactly holomorphic function in a neighbourhood of* ξ;

b) *For each sequence* $(\xi_n)_{n=1}^{\infty}$ *of elements of* U *which converges to some point in* ∂U *there exists* $f \in \mathscr{H}_c(U)$ *such that* $\sup_n |f(\xi_n)| = \infty$.

PROPOSITION 1.1. *Let* U *be a connected open subset of the Banach space* E *such that it is impossible to find two open connected sets* U_1, U_2 *such that* (1) $U_1 \cap U \supset U_2 \neq \varnothing$ *and* $U_1 \not\subset U$; (2) $\forall f \in \mathscr{H}_c(U)$ *there exists* $f_1 \in \mathscr{H}_c(U_1)$ *such that* $f|_{U_2} = f_1|_{U_2}$. *Then, for all compact* $K \subset U$, $\hat{K}_c$ *is* U-*bounded* (iff $\hat{K}_c$ *is compact*) *and* $d(\hat{K}_c, \partial U) = d(K, \partial U)$.

THEOREM 1.2. *Let* U *be a connected open subset of the Banach space* E *such that each* $\xi \in \partial U$ *has a fundamental system of neighbourhoods* $N_n(\xi)$ *such that* $U \cap N_n(\xi)$ *is connected for each n. Then the following properties are equivalent*:

a) *For each* $\xi \in \partial U$ *there exists* $f \in \mathscr{H}_{cb}(U)$ *which cannot be extended to a compactly holomorphic function of* U-*bounded type in a neighbourhood of* ξ.

b) *For each sequence* $(\xi_n)_{n=1}^{\infty}$ *of elements of* U *which converges to some point in* ∂U *there exists* $f \in \mathscr{H}_{cb}(U)$ *such that* $\sup_n |f(\xi_n)| = \infty$.

THEOREM 1.3. (Cartan–Thullen I). – *Let* E *be a Banach space. Let* U *be a connected open subset of* E *such that each* $\xi \in \partial U$ *has a fundamental system of neighbourhoods* $N_n(\xi)$ *such that* $U \cap N_n(\xi)$ *is connected for each n. Then the following properties are equivalent*:

(1) *For each* $\xi \in \partial U$ *there exists* $f \in \mathscr{H}_{cb}(U)$ *which cannot be extended to a compactly holomorphic function of* U-*bounded type in a neighbourhood of* ξ.

(2) *It is impossible to find two open connected sets* U_1, U_2 *such that*: (*i*) $U_1 \cap U \supset U_2 \neq \varnothing$ *and* $U_1 \not\subset U$; (*ii*) $\forall f \in \mathscr{H}_{cb}(U)$ *there exists* $f_1 \in \mathscr{H}_{cb}(U_1)$ *such that* $f|_{U_2} = f_1|_{U_2}$.

(3) U *is cb-holomorphically convex.*

(4) $\hat{B}_{cb}$ *is* U-*bounded for all* U-*bounded sets* $B \subset U$.

(5) *For each sequence* $(\xi_n)_{n=1}^{\infty}$ *of elements of* U *which converges to some point in* ∂U *there exists* $f \in \mathscr{H}_{cb}(U)$ *such that* $\sup_n |f(\xi_n)| = +\infty$.

THEOREM 1.4. *Let* U *be a connected open subset of the Banach space* E. *The following properties are equivalent*:

a) *For each* $\xi \in \partial U$ *and* $(\xi_n)_{n=1}^{\infty} \subset U$, $\xi_n \to \xi$ *as* $n \to \infty$ *there exists* $f \in \mathcal{H}_{cb}(U)$ $(f \in \mathcal{H}_c(U))$ *such that the radius of normal convergence of* f *about* ξ_n *tends to zero as* $n \to \infty$.

b) *For each sequence* $(\xi_n)_{n=1}^{\infty}$ *of elements of* U *which converges to some point in* ∂U *there exists* $f \in \mathcal{H}_{cb}(U)$ $(f \in \mathcal{H}_c(U))$ *such that* $\sup_n |f(\xi_n)| = +\infty$.

THEOREM 1.5. (Cartan-Thullen II). *Let* E *be a separable Banach space and* U *a connected open subset of* E. *The following properties are equivalent*:

(1) *For each* $\xi \in \partial U$ *there exists* $f \in \mathcal{H}_{cb}(U)$ *which cannot be extended to a compactly holomorphic function of* U*-bounded type in a neighbourhood of* ξ;

(2) *It is impossible to find two open connected sets* U_1, U_2 *such that*: (*i*) $U_1 \cap U \supset U_2 \neq \varnothing$ *and* $U_1 \not\subset U$, (*ii*) $\forall f \in \mathcal{H}_{cb}(U)$ *there exists* $f_1 \in \mathcal{H}_{cb}(U_1)$ *such that* $f|_{U_2} = f_1|_{U_2}$,

(3) $\hat{B}_{cb}$ *is* U*-bounded for all* U*-bounded sets* $B \subset U$;

(4) *There exists* $f \in \mathcal{H}_{cb}(U)$ *such that it is impossible to find two open connected subsets* U_1 *and* U_2 *of* E *satisfying the following conditions*: (*i*) $U \cap U_1 \supset U_2 \neq \varnothing$, $U_1 \not\subset U$. (*ii*) *there exists* $f_1 \in \mathcal{H}_{cb}(U_1)$ *such that* $f|_{U_2} = f_1|_{U_2}$.

2. CONSTRUCTION OF THE θ-ENVELOPE OF HOLOMORPHY FOR EVERY HOLOMORPHY TYPE

Notation: see [11] or [14].

DEFINITION 2.1. Let (X, φ) be a Riemann domain over a Banach space E. A function $f: X \to \mathbf{C}$ is holomorphic of type θ at $x \in X$ if there exists a neighbourhood V of x such that $V \sim \varphi(V)$ and a function $\psi \in \mathcal{H}_\theta(\varphi(V))$ such that $f = \psi \circ \varphi$ on V. We say that f is holomorphic of type θ on X if f is holomorphic of type θ at every $x \in X$.

Notation: $\mathcal{H}_\theta(X)$ = set of the holomorphic functions of type θ on X.

DEFINITION 2.2. Let (X, φ) and (X', φ') be Riemann domains over a Banach space E. A morphism u from (X, φ) over (X', φ') is a θ-holomorphic extension of (X, φ) if to every $f \in \mathcal{H}_\theta(X)$ we can associate one and only one $f' \in \mathcal{H}_\theta(X')$ such that $f = f' \circ u$.

DEFINITION 2.3. A θ-envelope of holomorphy of a Riemann domain (X, φ) is a Riemann domain $(\tilde{X}, \tilde{\varphi})$ and an extension $u: X \to \tilde{X}$ such that: if (X', φ') is a Riemann domain and $u': X \to X'$ is a θ-holomorphic extension of X, then there exists a θ-holomorphic extension of X', $\tilde{u}: X' \to \tilde{X}$ such that $u = \tilde{u} \circ u'$.

THEOREM 2.1. *Let* E *be a Banach space and* X *an open connected subset of* E. *Then there exists a* θ-*envelope of holomorphy of* X, *unique up to an isomorphism.*

THEOREM 5.2. *Every connected Riemann domain over a Banach space* E *has a* θ-*envelope of holomorphy, unique up to isomorphism.*

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